Non Local Conservation Laws in Bounded Domains

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Abstract

The well posedness for a class of non local systems of conservation laws in a bounded domain is proved and various stability estimates are provided. This construction is motivated by the modelling of crowd dynamics, which also leads to define a non local operator adapted to the presence of a boundary. Numerical integrations show that the resulting model provides qualitatively reasonable solutions.

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1 Introduction

Non local conservation laws are being developed to model various phenomena, such as the dynamics of crowd, see [12, 13, 14]; vehicular traffic, [11, 22]; supply chains, [5, 13]; granular materials, [2]; sedimentation phenomena, [9, 11]; and vortex dynamics, [6]. Often, these models are set in the whole space $\mathbb{R}^N$, although the physics might require their stating in domains with boundaries. Two difficulties typically motivate this simplification: the rigorous treatment of boundaries and boundary data in conservation laws is technically quite demanding, see [7, 16], and the very meaning of non local operators in the presence of a boundary is not straightforward, see [18, 22] for recent different approaches.

Furthermore, numerical methods for non local conservation laws are typically developed in the case of the Cauchy problem, i.e., on all of $\mathbb{R}$, see [3, 9, 11], or on all $\mathbb{R}^N$, see [1]. However, numerical integrations obviously refer to bounded domains and proper boundary conditions need to be singled out.

Below we tackle both the difficulties of a careful treatment of boundary conditions and of a proper use of non local operators in the presence of a boundary. While tackling these issues, we propose a rigorous construction yielding the well posedness of a class of non local conservation laws in bounded domains. Since the different equations are coupled through non local operators, we obtain the well posedness for a class of *systems* of conservation laws in *any* space dimension. The present construction is motivated by crowd dynamics and specific applications are explicitly considered.

Let I be a real interval and Ω be a bounded open subset of $\mathbb{R}^N$. We describe the movement of n populations, identified by their densities (or *occupancies*) $\rho \equiv (\rho^1, \ldots, \rho^n)$, through the following system of non local conservation laws:

$$\begin{cases} \partial_t \rho^i + \operatorname{div}\left[\rho^i \, V^i(t,x,\mathcal{J}^i\rho)\right] = 0 & (t,x) \in I \times \Omega \qquad i = 1,\ldots,n \\ \rho(t,\xi) = 0 & (t,\xi) \in I \times \partial\Omega \\ \rho(0,x) = \rho_o(x) & x \in \Omega \end{cases} \tag{1.1}$$

where $\rho_o \in \mathbf{L}^1(\Omega;\mathbb{R}^n)$ is a given initial datum and $\mathcal{J}^i$ is a non local operator, so that by the writing in the first equation of (1.1) we mean

$$\partial_t \rho^i(t,x) + \operatorname{div}\left[\rho^i(t,x)\, V^i\Big(t,x,\left(\mathcal{J}^i\rho(t)\right)(x)\Big)\right] = 0\,.$$

The choice of the zero boundary datum implies that no one can enter Ω from outside. Nevertheless, the usual definition of solution to conservation laws on domains with boundary, see [7, 27, 28], allows that individuals exit through the boundary.

The next section is devoted to the statement of the well posedness result. Section 3 deals with two specific sample applications to crowd dynamics. Proofs are left to the final sections 4 and 5.

2 Main Result

We set $\mathbb{R}_+ = [0,+\infty[$. The space dimension N, the number of equations n and the integer m are fixed throughout, with $N,n,m \geq 1$. We denote by I the time interval $\mathbb{R}_+$ or $[0,T]$, for a fixed $T>0$. Below, $B(x,\ell)$ for $x \in \mathbb{R}^N$ and $\ell > 0$ stands for the closed ball centred at x with radius ℓ.

Given the map $V\colon I\times\Omega\times\mathbb{R}^m \to \mathbb{R}^N$, where $(t,x,A) \in I\times\Omega\times\mathbb{R}^m$ and $\Omega \subset \mathbb{R}^N$, we set

$$\begin{aligned}
\nabla_x V(t,x,A) &= \left[\partial_{x_k} V_j(t,x,A)\right]_{\substack{j=1,\ldots,N\\ k=1,\ldots,N}} \in \mathbb{R}^{N\times N}\,,\\
\nabla_A V(t,x,A) &= \left[\partial_{A_l} V_j(t,x,A)\right]_{\substack{j=1,\ldots,N\\ l=1,\ldots,m}} \in \mathbb{R}^{N\times m},\\
\nabla_{x,A} V(t,x,A) &= \left[\nabla_x V(t,x,A) \quad \nabla_A V(t,x,A)\right] \in \mathbb{R}^{N\times(N+m)}\,,\\
\left\|V(t)\right\|_{\mathbf{C}^2(\Omega\times\mathbb{R}^m;\mathbb{R}^N)} &= \left\|V(t)\right\|_{\mathbf{L}^\infty(\Omega\times\mathbb{R}^m;\mathbb{R}^N)} + \left\|\nabla_{x,A} V(t)\right\|_{\mathbf{L}^\infty(\Omega\times\mathbb{R}^m;\mathbb{R}^{N\times(N+m)})}\\
&\quad + \left\|\nabla^2_{x,A} V(t)\right\|_{\mathbf{L}^\infty(\Omega\times\mathbb{R}^m;\mathbb{R}^{N\times(N+m)\times(N+m)})}\,.
\end{aligned}$$

For $\rho \in \mathbf{L}^\infty(\Omega;\mathbb{R}^n)$, we also denote $\mathrm{TV}\,(\rho) = \sum_{i=1}^n \mathrm{TV}\,(\rho^i)$.

We pose the following assumptions:

($\boldsymbol{\Omega}$) $\Omega \subset \mathbb{R}^N$ is non empty, open, connected, bounded and with $\mathbf{C}^2$ boundary $\partial\Omega$.

(V) For $i=1,\ldots,n$, $V^i \in (\mathbf{C}^0 \cap \mathbf{L}^\infty)(I\times\Omega\times\mathbb{R}^m;\mathbb{R}^N)$; for all $t \in I$, $V^i(t) \in \mathbf{C}^2(\Omega\times\mathbb{R}^m;\mathbb{R}^N)$ and $\left\|V^i(t)\right\|_{\mathbf{C}^2(\Omega\times\mathbb{R}^m;\mathbb{R}^N)}$ is bounded uniformly in t and i, i.e., there exists a positive constant $\mathcal{V}$ such that $\left\|V^i(t)\right\|_{\mathbf{C}^2(\Omega\times\mathbb{R}^m;\mathbb{R}^N)} \leq \mathcal{V}$ for all $t \in I$ and all $i=1,\ldots,n$.

(J) For $i=1,\ldots,n$, $\mathcal{J}^i\colon \mathbf{L}^1(\Omega;\mathbb{R}^n) \to \mathbf{C}^2(\Omega;\mathbb{R}^m)$ is such that there exists a positive K and a weakly increasing map $\mathcal{K} \in \mathbf{L}^\infty_{\mathbf{loc}}(\mathbb{R}_+;\mathbb{R}_+)$ such that

(J.1) for all $r \in \mathbf{L}^1(\Omega;\mathbb{R}^n)$,

$$\begin{aligned}
\left\|\mathcal{J}^i(r)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^m)} &\leq K\,\|r\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\,,\\
\left\|\nabla_x\mathcal{J}^i(r)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{m\times N})} &\leq K\,\|r\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\,,\\
\left\|\nabla^2_x\mathcal{J}^i(r)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{m\times N\times N})} &\leq \mathcal{K}\left(\|r\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\right)\,\|r\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\,.
\end{aligned}$$

(J.2) for all r_1, $r_2 \in \mathbf{L}^1(\Omega;\mathbb{R}^n)$

$$\begin{aligned}
\left\|\mathcal{J}^i(r_1) - \mathcal{J}^i(r_2)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^m)} &\leq K\,\|r_1 - r_2\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\,,\\
\left\|\nabla_x\left(\mathcal{J}^i(r_1) - \mathcal{J}^i(r_2)\right)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{m\times N})} &\leq \mathcal{K}\left(\|r_1\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\right)\,\|r_1 - r_2\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\,.
\end{aligned}$$

Throughout, $\mathcal{O}(1)$ denotes a constant dependent only on norms of the functions in the assumptions above, in particular it is independent of time.

Recall that if Ω satisfies **($\boldsymbol{\Omega}$)**, then it also enjoys the *interior sphere condition with radius* $r > 0$, in the sense that for all $\xi \in \partial\Omega$, there exists $x \in \Omega$ such that $B(x, \ell) \subseteq \Omega$ and $\xi \in \partial B(x, \ell)$ see [19, Section 6.4.2] and [20, Section 3.2].

In conservation laws, boundary conditions are enforced along the boundary only where characteristic velocities enter the domain, so that admissible jump discontinuities between boundary data and boundary trace of the solution have to be selected. This is provided by the following definition, based on *regular entropy solutions*, see [27, Definition 3.3], [28, Definition 2.2] and Definition 4.2 below.

Definition 2.1. *A map* $\rho \in \mathbf{C}^0\left(I; \mathbf{L}^1(\Omega; \mathbb{R}^n)\right)$ *is a* solution *to* (1.1) *whenever, setting* $u^i(t,x) = V^i\left(t, x, \left(\mathcal{J}^i \rho(t)\right)(x)\right)$, *for* $i = 1, \ldots, n$, *the map* ρ^i *is a regular entropy solution to*

$$\begin{cases} \partial_t \rho^i + \operatorname{div}\left[\rho^i\, u^i(t,x)\right] = 0 & (t,x) \in I \times \Omega\,, \\ \rho^i(t,\xi) = 0 & (t,\xi) \in I \times \partial\Omega\,, \\ \rho^i(0,x) = \rho_o^i(x) & x \in \Omega\,. \end{cases} \tag{2.1}$$

We are now ready to state the main result of this paper.

Theorem 2.2. *Let* **($\boldsymbol{\Omega}$)** *hold. Fix* V *satisfying* **(V)** *and* $\mathcal{J}$ *satisfying* **(J)**. *Then:*

(1) For any $\rho_o \in (\mathbf{L}^\infty \cap \mathbf{BV})(\Omega; \mathbb{R}^n)$, *there exists a unique* $\rho \in \mathbf{L}^\infty(I \times \Omega; \mathbb{R}^n)$ *solving* (1.1) *in the sense of Definition 2.1.*

(2) For any $\rho_o \in (\mathbf{L}^\infty \cap \mathbf{BV})(\Omega; \mathbb{R}^n)$ *and for any* $t \in I$,

$$\begin{aligned} \left\|\rho(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} &\le \|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} \\ \left\|\rho(t)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)} &\le \|\rho_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)} \exp\left(t\,\mathcal{V}\left(1 + K\,\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\right)\right) \\ \operatorname{TV}\left(\rho(t)\right) &\le \exp\left(t\,\mathcal{V}\left(1 + K\,\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\right)\right) \\ &\quad \times \left[\mathcal{O}(1) n\,\|\rho_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)} + \operatorname{TV}(\rho_o) + n\,t\,\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\,\mathcal{V}\right. \\ &\quad \left.\times\left(1 + \|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\left(K + K^2\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} + \mathcal{K}\Big(\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\Big)\right)\right)\right]. \end{aligned}$$

(3) For any $\rho_o \in (\mathbf{L}^\infty \cap \mathbf{BV})(\Omega; \mathbb{R}^n)$ *and for any* $t, s \in I$,

$$\left\|\rho(t) - \rho(s)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} \le \operatorname{TV}\left(\rho\left(\max\{t,s\}\right)\right)|t-s|.$$

(4) For any initial data $\rho_o, \tilde\rho_o \in (\mathbf{L}^\infty \cap \mathbf{BV})(\Omega; \mathbb{R}^n)$ *and for any* $t \in I$, *calling* ρ *and* $\tilde\rho$ *the corresponding solutions to* (1.1),

$$\left\|\rho(t) - \tilde\rho(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} \le e^{\mathcal{L}(t)}\,\|\rho_o - \tilde\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)},$$

where $\mathcal{L}(t) > 0$ *depends on* **($\boldsymbol{\Omega}$)**, **(V)**, **(J)** *and on*

$$R = \max\left\{\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)},\, \|\tilde\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)},\, \|\rho_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)},\, \|\tilde\rho_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)},\, \operatorname{TV}(\rho_o),\, \operatorname{TV}(\tilde\rho_o)\right\}.$$

(5) *Fix $\rho_o \in (\mathbf{L}^\infty \cap \mathbf{BV})(\Omega;\mathbb{R}^n)$. Let $\tilde V$ satisfy* **(V)** *with the same constant $\mathcal{V}$. Call ρ and $\tilde\rho$ the solutions to problem* (1.1) *corresponding respectively to the choices V and $\tilde V$. Then, for any $t \in I$,*

$$\left\|\rho(t) - \tilde\rho(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} \le \mathcal{C}(t) \int_0^t \left\|V(s) - \tilde V(s)\right\|_{\mathbf{C}^1(\Omega\times\mathbb{R}^m;\mathbb{R}^{nN})} \mathrm{d}s$$

where $\mathcal{C}$ depends on **(Ω)**, **(V)**, **(J)** *and on the initial datum, see* (4.33).

(6) *For $i = 1, \ldots, n$, if $\rho_o^i \ge 0$ a.e. in Ω, then $\rho^i(t) \ge 0$ a.e. in Ω for all $t \in I$.*

Section 4 is devoted to the proof of the theorem above. Here, we underline that the total variation estimate in (2) is qualitatively different from the analogous one in the case of no boundary, see Remark 4.5.

3 The Case of Crowd Dynamics

The above analytic results are motivated also by their applicability to equations describing the motion of a crowd, identified through its time and space dependent density $\rho = \rho(t,x)$. Various macroscopic crowd dynamics models based on non local conservation laws were recently considered, see for instance [12, 13, 14], as well as [1, Section 3.1]. Therein, typically, non local interactions among individuals are described through space convolution terms like $\rho(t) * \eta$, for a suitable averaging kernel η. We refer to [17] for a different approach and to [8] for a recent review on the modelling of crowd dynamics.

Due to the absence of well posedness results in bounded domains, none of the results cited above considers the presence of boundaries. On the one hand, the choice of the crowd velocity may well encode the presence of boundaries but, on the other hand, the visual horizon of each individual should definitely not neglect the presence of the boundary. With this motivation, below we introduce a non local operator consistent with the presence of boundaries and show how the theoretical results above allow to formulate equations where each individual's horizon is affected by the presence of the walls.

To this aim, we use the following modification of the usual convolution product

$$(\rho *_\Omega \eta)(x) = \frac{1}{z(x)} \int_\Omega \rho(y)\, \eta(x-y)\, \mathrm{d}y\ , \qquad \text{where} \tag{3.1}$$

$$z(x) = \int_\Omega \eta(x-y)\, \mathrm{d}y\ . \tag{3.2}$$

A reasonable assumption on the kernel η is:

(η) $\eta(x) = \tilde\eta(\|x\|)$, where $\tilde\eta \in \mathbf{C}^2(\mathbb{R}_+;\mathbb{R})$, $\operatorname{spt}\tilde\eta = [0,\ell_\eta]$, where $\ell_\eta > 0$, $\tilde\eta' \le 0$ and $\int_{\mathbb{R}^N} \eta(\xi)\,\mathrm{d}\xi = 1$.

In other words, $(\rho *_\Omega \eta)(x)$ is an average of the crowd density ρ in Ω around x. Note also that $\rho *_\Omega \eta$ is well defined by (3.1): indeed, under assumptions **(Ω)** and **(η)**, z may not vanish in Ω, see Lemma 5.1. As a side remark, note that **(η)** ensures $\eta \ge 0$.

We investigate the properties of the non local operator defined through (3.1)–(3.2).

Lemma 3.1. *Let Ω satisfy* **(Ω)**, *η satisfy* **(η)** *and $\rho \in \mathbf{L}^\infty(\Omega;\mathbb{R}_+)$. Then,*

$$(\rho *_\Omega \eta) \in \mathbf{C}^2(\Omega;\mathbb{R}_+) \quad \textit{and} \quad (\rho *_\Omega \eta)(x) \in \Big[\operatorname*{ess\,inf}_{B(x,\ell_\eta)\cap\Omega} \rho,\ \operatorname*{ess\,sup}_{B(x,\ell_\eta)\cap\Omega} \rho\Big]$$

so that, in particular, $(\rho *_\Omega \eta)(\Omega) \subseteq [0, \|\rho\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}]$.

The proof is in Section 5, where other properties of the modified convolution (3.1)–(3.2) are proved.

As a sample of the possible applications of Theorem 2.2 to crowd dynamics, we consider below two specific situations, where we set $N = 2$, write $x \equiv (x_1, x_2)$ for the spatial coordinate and denote $\partial_1 = \partial_{x_1}$, $\partial_2 = \partial_{x_2}$, The numerical integrations below are obtained through a suitable adaptation of the Lax–Friedrichs method, on the basis of [1, 3], adapted as suggested in [10, Formula (14)] to reduce the effects of the numerical viscosity.

For further results on crowd modelling, see for instance [12, 24, 25] and the references therein.

3.1 Evacuation from a Room

We now use (1.1) to describe the evacuation of a region, say Ω. To this aim, consider the equation:

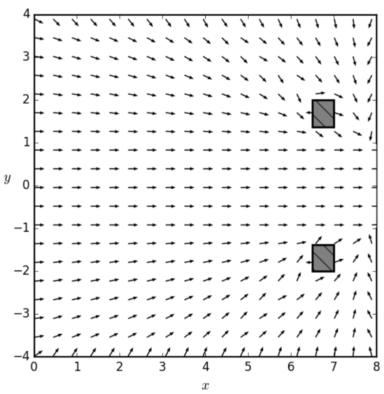
$$\partial_t \rho + \operatorname{div} \left[\rho \; v(\rho *_{\Omega} \eta_1) \left(w(x) - \beta \frac{\nabla(\rho *_{\Omega} \eta_2)}{\sqrt{1 + \left\| \nabla(\rho *_{\Omega} \eta_2) \right\|^2}} \right) \right] = 0 \,. \tag{3.3}$$

Here, each individual adjusts her/his speed according to the average population density around her/him, according to the function v, which is $\mathbf{C}^2$, bounded and non increasing. The velocity direction of each individual is given by the fixed $\mathbf{C}^2$ vector field w, which essentially describes some sort of *natural* path to the exit, the exit being the portion of $\partial\Omega$ where w points outwards of Ω. This direction is then adjusted by the non local term $-\beta \, \nabla(\rho *_{\Omega} \eta_2) \Big/ \sqrt{1 + \left\| \nabla(\rho *_{\Omega} \eta_2) \right\|}$, which describes the tendency of avoiding regions with high (average) density gradient, see [12, 14].

Lemma 3.2. *Let Ω satisfy* **(Ω)**. *Assume that $v \in \mathbf{C}^2(\mathbb{R}_+; \mathbb{R}_+)$ and $w \in \mathbf{C}^2(\Omega; \mathbb{R}^2)$ are bounded in $\mathbf{C}^2$. If moreover η_1, η_2 satisfy* **($\boldsymbol{\eta}$)** *with η_2 of class $\mathbf{C}^3$, then equation* (3.3) *fits into* (1.1), **(V)** *and* **(J)** *hold, so that Theorem 2.2 applies.*

The proof is deferred to Section 5.

As a specific example we consider a square room, say Ω, with a door D, with $D \subseteq \partial\Omega$, and two columns each of size 0.5×0.625, placed near to the door, symmetrically as the grey rectangles in the figure in (3.4). We also set



$$\begin{aligned} \Omega &= [0, 8] \times [-4, 4] \\ D &= \{8\} \times [-1, 1] \\ \tilde{\eta}_i(\xi) &= \frac{315}{128\, \pi \, {l_i}^{18}} \, ({l_i}^4 - \xi^4)^4 \, \chi_{[0, l_i]}(\xi) \\ v(r) &= 2 \, \min\left\{1, \max\left\{0, (1 - (r/4)^3)^3\right\}\right\} \\ w(x) &= \text{ see the figure here on the left,} \\ l_1 &= 0.625 \,, \quad l_2 = 1.5 \,, \quad \beta = 0.6 \,. \end{aligned} \tag{3.4}$$

The vector field $w = w(x)$ is obtained as a sum of the unit vector tangent to the geodesic from x to the door and a discomfort vector field with maximal intensity along the walls. The numerical integration corresponding to a locally constant initial datum is displayed in Figure 1. The solution displays a realistic behaviour, with queues being formed behind the obstacles. For further details on the modelling and numerical issues related to (1.1)–(3.3)–(3.4), we refer to [15].

3.2 Two Ways Movement along a Corridor

The validity of Theorem 2.2 also for systems of equations allows to consider the case of interacting populations. A case widely considered in the literature, see for instance [1, 12, 14, 17, 24] and the references in [8], is that of two groups of pedestrians heading in opposite directions along

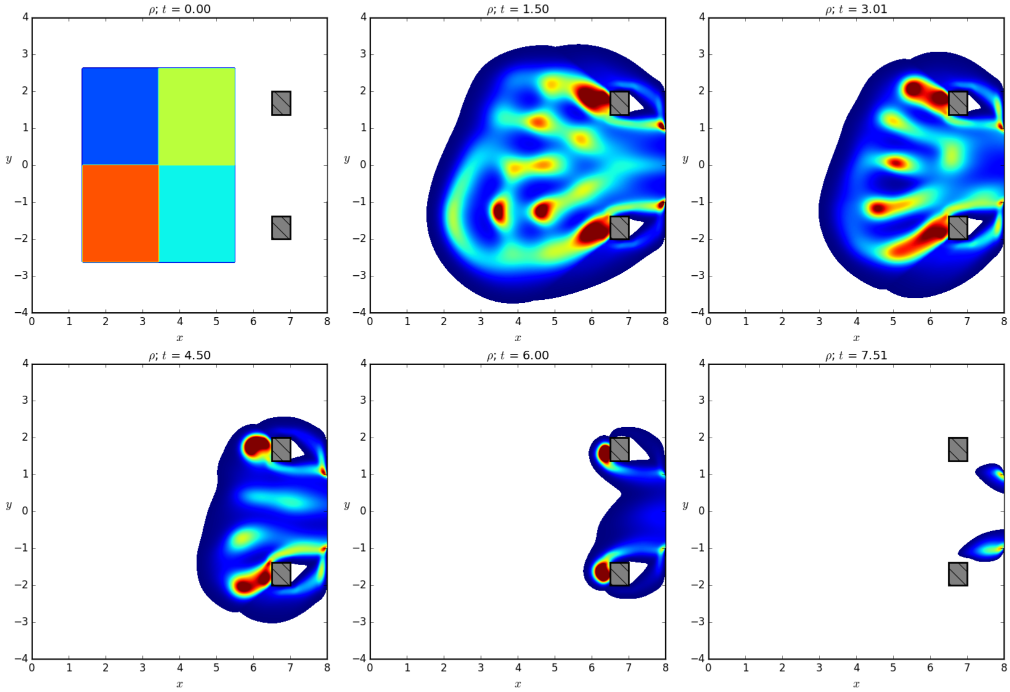


Figure 1: Plot of the level curves of the solution to (1.1)–(3.3)–(3.4), computed numerically at the times $t = 0,\ 1.5,\ 3.0,\ 4.5,\ 6.0,\ 7.5$, corresponding to the initial data in the top left figure, consisting of 5, 14, 9 and 20 people (clockwise starting from the top left) in the 4 quadrants displayed in the first figure. In this integration, the mesh sizes are $dx = dy = 0.03125$.

a corridor, say Ω, with exits, say D, on each of its sides. With the notation in Section 2, this amounts to set $N = 2$, $n = 2$ and to

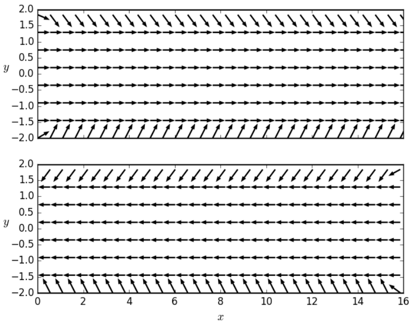
$$\begin{cases} \partial_t \rho^1 + \operatorname{div}\left[\rho^1 v^1\left((\rho^1+\rho^2) *_\Omega \eta_1^{11}\right)\left(w^1(x) - \dfrac{\beta_{11}\nabla(\rho^1 *_\Omega \eta_2^{11})}{\sqrt{1+\left\|\nabla(\rho^1 *_\Omega \eta_2^{11})\right\|^2}} - \dfrac{\beta_{12}\nabla(\rho^2 *_\Omega \eta_2^{12})}{\sqrt{1+\left\|\nabla(\rho^2 *_\Omega \eta_2^{12})\right\|^2}}\right)\right] = 0, \\ \partial_t \rho^2 + \operatorname{div}\left[\rho^2 v^2\left((\rho^1+\rho^2) *_\Omega \eta_1^{22}\right)\left(w^2(x) - \dfrac{\beta_{21}\nabla(\rho^1 *_\Omega \eta_2^{21})}{\sqrt{1+\left\|\nabla(\rho^1 *_\Omega \eta_2^{21})\right\|^2}} - \dfrac{\beta_{22}\nabla(\rho^2 *_\Omega \eta_2^{22})}{\sqrt{1+\left\|\nabla(\rho^2 *_\Omega \eta_2^{22})\right\|^2}}\right)\right] = 0. \end{cases} \tag{3.5}$$

The various terms in the expressions above are straightforward extensions of their analogues in (3.3). For instance, in view of (3.1)–(3.2), $v^i = v^i\left((\rho^1+\rho^2) *_\Omega \eta_1^{ii}\right)$ describes how the maximal speed of the population i at a point x depends on the average total density of $\rho^1+\rho^2$ in Ω around x. Similarly, the term $-\beta_{ij}\,\nabla(\rho^i *_\Omega \eta_2^{ij}) \Big/ \sqrt{1+\left\|\rho^i *_\Omega \eta_2^{ij}\right\|^2}$ describes the tendency of individuals of the i-th population to avoid increasing values of the average density of the j-th population, in the same spirit of the similar term in (3.3).

Lemma 3.3. *Let Ω satisfy* **(Ω)**. *Assume that $v^1, v^2 \in \mathbf{C}^2(\mathbb{R}_+;\mathbb{R}_+)$ and $w^1, w^2 \in \mathbf{C}^2(\Omega;\mathbb{R}^2)$ are bounded in $\mathbf{C}^2$. If moreover η_1^{ii}, η_2^{ij} satisfy (η) with η_2^{ij} of class $\mathbf{C}^3$ for $i,j = 1,2$, then equation* (3.5) *fits into* (1.1)*,* **(V)** *and* **(J)** *hold, so that Theorem 2.2 applies.*

The proof is deferred to Section 5.

A qualitative picture of the possible solutions to (1.1)–(3.5) is obtained through the following numerical integration, corresponding to the choices



$$
\begin{aligned}
\Omega &= [0,16] \times [-2,2]\,, \qquad D = \{0,16\} \times [-2,2] \\
\tilde\eta_l^{ij}(\xi) &= \frac{315}{128\,\pi\,(l_i^{ij})^2}\,(1-(\xi/l_i^{ij})^4)^4\,\chi_{[0,l_i^{ij}]}(\xi) \\
v^1(r) &= \min\left\{1, \max\left\{0, (1-(r/4.5)^3)^3\right\}\right\} \\
v^2(r) &= 1.5\ \min\left\{1, \max\left\{0, (1-(r/4.5)^3)^3\right\}\right\} \\
w^1(x) &= \text{ see the figure here on the left, top} \\
w^2(x) &= \text{ see the figure here on the left, bottom} \\
l_1^{ii} &= 0.1875\,, \quad l_2^{ij} = 0.5\,, \quad \beta_{ii} = 0.2\,, \quad \beta_{ij} = 0.5\,.
\end{aligned}
\tag{3.6}
$$

for $i,j = 1,2$, see Figure 2. Note the complex dynamics arising due to the formation of regions with high density. This description is consistent with the typical *self organization* of crowd motions,

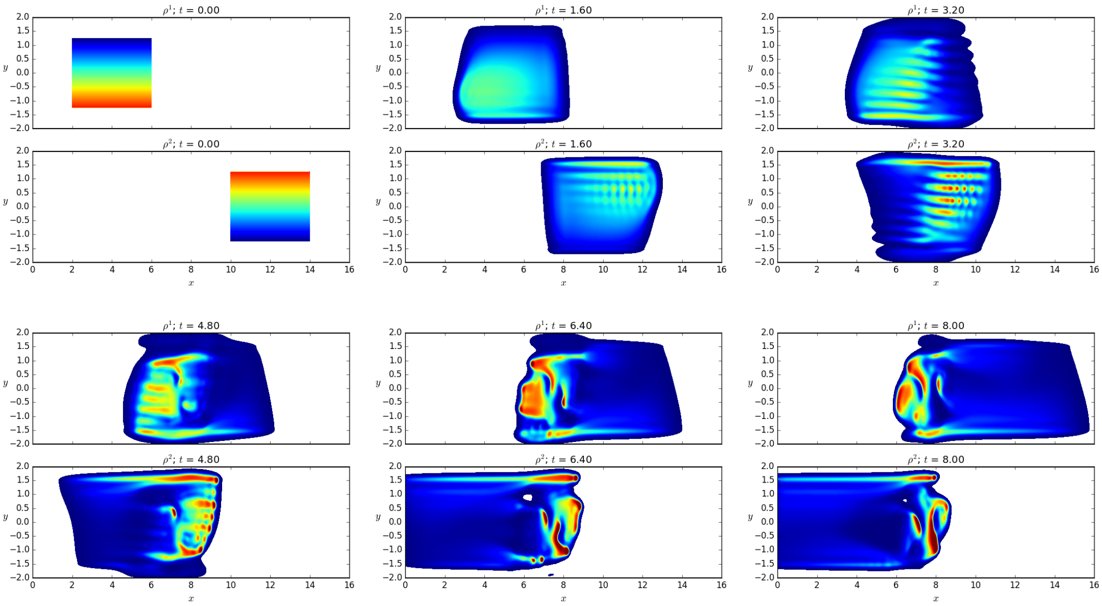


Figure 2: Plot of the level curves of the solution to (1.1)–(3.5)–(3.6), computed numerically at the times $t = 0,\ 1.6,\ 3.2,\ 4.8,\ 6.4,\ 8.0$. First and third rows refer to ρ^1, while the second and fourth one to ρ^2. The initial datum varies linearly along the y coordinate between 0 and 4. In this integration, the mesh sizes are $dx = dy = 0.015625$.

see [23, 24]: queues consisting of pedestrian walking in the same direction are formed, in particular at time 3.20.

4 Proofs Related to Section 2

We recall the basic properties of the following (local) IBVP

$$
\begin{cases}
\partial_t r + \operatorname{div}\left[r\ u(t,x)\right] = 0 & (t,x) \in I \times \Omega \\
r(t,\xi) = 0 & (t,\xi) \in I \times \partial\Omega \\
r(0,x) = r_o(x) & x \in \Omega\,,
\end{cases}
\tag{4.1}
$$

where we assume that

(u) $u\colon I \times \Omega \to \mathbb{R}^N$ is such that $u \in (\mathbf{C}^0 \cap \mathbf{L}^\infty)(I \times \Omega; \mathbb{R}^N)$, for all $t \in I$, $u(t) \in \mathbf{C}^2(\Omega; \mathbb{R}^N)$ and $\left\|u(t)\right\|_{\mathbf{C}^2(\Omega;\mathbb{R}^N)}$ is uniformly bounded in I.

We refer to [27] for a comparison among various definitions of solutions to (4.1). Recall the concept of *RE–solutions*, which first requires an extension of [26, Chapter 2, Definition 7.1]. Note that, although the equation in (4.1) is linear, jump discontinuities may well arise between the solution and the datum assigned along the boundary.

Definition 4.1 ([28, Definition 2])**.** *The pair $(H, Q) \in \mathbf{C}^2(\mathbb{R}^2; \mathbb{R}) \times \mathbf{C}^2(I \times \overline{\Omega} \times \mathbb{R}^2; \mathbb{R}^N)$ is called a* boundary entropy-entropy flux pair *for the flux $f(t, x, r) = r\, u(t, x)$ if:*

i) for all $w \in \mathbb{R}$ the function $z \mapsto H(z, w)$ is convex;

ii) for all $t \in I$, $x \in \overline{\Omega}$ and z, $w \in \mathbb{R}$, $\partial_z Q(t, x, z, w) = \partial_z H(z, w)\, u(t, x)$;

iii) for all $t \in I$, $x \in \overline{\Omega}$ and $w \in \mathbb{R}$, $H(w, w) = 0$, $Q(t, x, w, w) = 0$ and $\partial_z H(w, w) = 0$.

Note that if H is as above, then $H \geq 0$.

Definition 4.2 ([27, Definition 3.3])**.** *A* Regular Entropy solution *(RE-solution) to the initial–boundary value problem* (4.1) *on I is a map $r \in \mathbf{L}^\infty (I \times \Omega; \mathbb{R})$ such that for any boundary entropy-entropy flux pair (H, Q), for any $k \in \mathbb{R}$ and for any test function $\varphi \in \mathbf{C}^1_c(\mathbb{R} \times \mathbb{R}^N; \mathbb{R}_+)$*

$$
\begin{aligned}
&\int_I \int_\Omega \Big[H\left(r(t,x),k\right) \partial_t \varphi(t,x) + Q\left(t,x,r(t,x),k\right) \cdot \nabla\varphi(t,x)\Big] \mathrm{d}x\, \mathrm{d}t \\
&- \int_I \int_\Omega \partial_1 H\left(r(t,x),k\right)\, r(t,x)\, \operatorname{div} u(t,x)\, \varphi(t,x)\, \mathrm{d}x\, \mathrm{d}t \\
&+ \int_I \int_\Omega \operatorname{div} Q\left(t,x,r(t,x),k\right)\, \varphi(t,x)\, \mathrm{d}x\, \mathrm{d}t \\
&+ \int_\Omega H\left(r_o(x),k\right)\, \varphi(0,x)\, \mathrm{d}x + \|u\|_{\mathbf{L}^\infty(I\times\Omega;\mathbb{R}^N)} \int_I \int_{\partial\Omega} H\left(0,k\right)\, \varphi(t,\xi)\, \mathrm{d}\xi\, \mathrm{d}t \geq 0.
\end{aligned}
\tag{4.2}
$$

Lemma 4.3. *Let* **(Ω)** *and* **(u)** *hold. Assume $r_o \in (\mathbf{L}^\infty \cap \mathbf{BV})(\Omega; \mathbb{R})$. For $(t_o, x_o) \in I \times \Omega$, introduce the map*

$$
\begin{array}{rccc}
X(\,\cdot\,; t_o, x_o)\colon & I(t_o,x_o) & \to & \Omega \\
& t & \to & X(t; t_o, x_o)
\end{array}
\qquad \textit{solving} \qquad
\left\{
\begin{array}{l}
\dot{x} = u(t,x) \\
x(t_o) = x_o\,,
\end{array}
\right.
\tag{4.3}
$$

$I(t_o, x_o)$ being the maximal interval where a solution to the Cauchy problem above is defined. The map r defined by

$$
r(t,x) = \left\{
\begin{array}{ll}
r_o\left(X(0;t,x)\right) \exp\left(-\displaystyle\int_0^t \operatorname{div} u\left(\tau, X(\tau;t,x)\right) \mathrm{d}\tau\right) & x \in X(t;0,\Omega) \\
0 & x \in X(t; [0,t[\,, \partial\Omega)
\end{array}
\right.
\tag{4.4}
$$

is a RE–solution to (4.1)*. Moreover, $r\colon [0,T] \to (\mathbf{L}^\infty \cap \mathbf{BV})(\Omega; \mathbb{R})$ is $\mathbf{L}^1$–continuous.*

Proof. We first regularise the initial datum, using [4, Theorem 1], see also [21, Formula (1.8) and Theorem 1.17]: for $h \in \mathbb{N} \setminus \{0\}$, there exists a sequence $\tilde{r}_h \in \mathbf{C}^\infty(\Omega; \mathbb{R})$ such that

$$
\lim_{h\to+\infty} \|\tilde{r}_h - r_o\|_{\mathbf{L}^1(\Omega;\mathbb{R})} = 0, \qquad \|\tilde{r}_h\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \leq \|r_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \quad \text{and} \quad \lim_{h\to\infty} \operatorname{TV}(\tilde{r}_h) = \operatorname{TV}(r_o).
$$

Let $\Phi_h \in \mathbf{C}^3_c(\mathbb{R}^N; [0,1])$ be such that $\Phi_h(\xi) = 1$ for all $\xi \in \partial\Omega$, $\Phi_h(x) = 0$ for all $x \in \Omega$ with $B(x, 1/h) \subseteq \Omega$ and $\|\nabla \Phi_h\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^N)} \leq 2h$. Let

$$
r_o^h(x) = \left(1 - \Phi_h(x)\right)\, \tilde{r}_h(x) \quad \text{ for all } x \in \overline{\Omega},
\tag{4.5}
$$

so that $r_o^h \in \mathbf{C}^3(\Omega;\mathbb{R})$. By construction, $\lim_{h\to+\infty} \left\| r_o^h - r_o \right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} = 0$. Moreover, $r_o^h(\xi) = 0$ for all $\xi \in \partial\Omega$ and $h \in \mathbb{N}\setminus\{0\}$, and the following uniform bounds hold

$$\left\| r_o^h \right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \le \left\| r_o \right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}, \tag{4.6}$$

$$\mathrm{TV}\,(r_o^h) \le \mathcal{O}(1) \left\| r_o \right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \mathrm{TV}\,(r_o). \tag{4.7}$$

Using the sequence r_o^h, define the corresponding sequence r_h according to (4.4). Obviously, each r_h is a strong solution to (4.1). By [27, Proposition 6.2], each r_h is also a RE–solution to (4.1)

Let r be defined as in (4.4). It is clear that r_h converges to r in $\mathbf{L}^1$. Since Definition 4.2 is stable under $\mathbf{L}^1$ convergence, see [26, 27], we obtain that r is a RE–solution to (4.1).

The continuity in time of r follows from the continuity in time of r_h and the fact that r is the uniform limit of r_h. □

The following Lemma extends to the case of the IBVP the results in [13, Lemma 5.1, Corollary 5.2 and Lemma 5.3]. Note that, due to the presence of the boundary, this extension needs some care, see Remark 4.5.

Lemma 4.4. *Let* **(Ω)** *and* **(u)** *hold. Assume $r_o \in (\mathbf{L}^\infty \cap \mathbf{BV})(\Omega;\mathbb{R})$. Then, the solution r to* (4.1) *is such that $r \in \mathbf{C}^{0,1}(I;\mathbf{L}^1(\Omega;\mathbb{R}))$ and for all t, $s \in I$,*

$$\left\| r(t) \right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \le \left\| r_o \right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \tag{4.8}$$

$$\left\| r(t) \right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \le \left\| r_o \right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} e^{\left\| \operatorname{div} u \right\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}} \tag{4.9}$$

$$\begin{aligned} \mathrm{TV}\big(r(t)\big) \le\ & \exp\left(\int_0^t \left\| \nabla u\,(\tau) \right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{N\times N})} \mathrm{d}\tau \right) \Bigg(\mathcal{O}(1) \left\| r_o \right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \\ & + \mathrm{TV}\,(r_o) + \left\| r_o \right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \int_0^t \left\| \nabla \operatorname{div} u(\tau) \right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^N)} \mathrm{d}\tau \Bigg), \end{aligned} \tag{4.10}$$

$$\left\| r(t) - r(s) \right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \le \mathrm{TV}\left(r\left(\max\{t,s\}\right)\right) |t-s|\,. \tag{4.11}$$

If also $\tilde r_o \in (\mathbf{L}^\infty \cap \mathbf{BV})(\Omega;\mathbb{R})$ and $\tilde r$ is the corresponding solution to (4.1)*, for all $t \in I$,*

$$\left\| r(t) - \tilde r(t) \right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \le \left\| r_o - \tilde r_o \right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\,. \tag{4.12}$$

Proof. The proofs of (4.8) and (4.9) directly follow from (4.4). In particular, to get (4.8), exploit the change of variable $y = X(0;t,x)$, so that $x = X(t;0,y)$, see [13, § 5.1]. Note that if $x \in X(t;0,\Omega)$ then $y \in X\left(0;t,X(t;0,\Omega)\right) \subseteq \Omega$. Denote the Jacobian of this change of variable by $J(t,y) = \det\left(\nabla_y X(t;0,y)\right)$. Then J solves

$$\frac{\mathrm{d}J(t,y)}{\mathrm{d}t} = \operatorname{div} u\left(t, X(t;0,y)\right) J(t,y) \quad \text{with} \quad J(0,y) = 1.$$

Hence, $J(t,y) = \exp\left(\int_0^t \operatorname{div} u\left(\tau, X(\tau;0,y)\right) \mathrm{d}\tau\right)$, which implies $J(t,y) > 0$ for $t \in [0,T]$ and $y \in \Omega$.

To prove (4.10), regularise the initial datum r_o as in the proof of Lemma 4.3: $r_o^h \in \mathbf{C}^3(\Omega;\mathbb{R})$ converges to r_o in $\mathbf{L}^1(\Omega;\mathbb{R})$, $r_o^h(\xi) = 0$ for all $\xi \in \partial\Omega$ and (4.6)–(4.7) hold.

Using the sequence r_o^h, define according to (4.4) the corresponding sequence r_h of solutions to (4.1). Observe that $r_h(t) \in \mathbf{C}^1(\Omega;\mathbb{R})$ for every $t \in [0,T]$. Proceed similarly to the proof of [13, Lemma 5.4]: differentiate the solution to (4.3) with respect to the initial point, that is

$$\nabla_x X(\tau;t,x) = \mathbf{Id} + \int_t^\tau \nabla_x u\left(t, X(s;t,x)\right) \nabla_x X(s;t,x)\,\mathrm{d}s\,,$$

$$\left\| \nabla_x X(\tau;t,x) \right\| \le 1 + \int_\tau^t \left\| \nabla_x u\left(t, X(s;t,x)\right) \right\| \left\| \nabla_x X(s;t,x) \right\| \mathrm{d}s\,,$$

since $\tau \in [0,t]$, so that, applying Gronwall Lemma,

$$\left\|\nabla_x X(\tau;t,x)\right\| \le \exp\left(\int_\tau^t \left\|\nabla_x u(s)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{N\times N})} \mathrm{d}s\right).$$

By (4.4) and the properties of r_o^h, the gradient of $r_h(t)$ is well defined (and continuous) on Ω: in particular,

$$\begin{aligned}\nabla r_h(t,x) = {} & \exp\left(\int_0^t -\operatorname{div} u\left(\tau, X(\tau;t,x)\right) \mathrm{d}\tau\right)\left(\nabla r_o^h\left(X(0;t,x)\right)\, \nabla_x X(0;t,x)\right.\\ & \left. - r_o^h\left(X(0;t,x)\right) \int_0^t \nabla \operatorname{div} u\left(\tau, X(\tau;t,x)\right)\, \nabla_x X(\tau;t,x)\, \mathrm{d}\tau\right).\end{aligned}$$

Hence, for every $t \in I$, using again the change of variable described at the beginning of the proof,

$$\begin{aligned}\left\|\nabla r_h(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^N)} \le {} & \exp\left(\int_0^t \left\|\nabla u\,(\tau)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{N\times N})} \mathrm{d}\tau\right)\\ & \times\left(\int_\Omega \left|\nabla r_o^h(x)\right| \mathrm{d}x + \left\|r_o^h\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \int_0^t \left\|\nabla \operatorname{div} u(\tau)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^N)} \mathrm{d}\tau\right).\end{aligned} \tag{4.13}$$

Let r be defined as in (4.4): clearly, $r_h \to r$ in $\mathbf{L}^1(\Omega;\mathbb{R})$. Due to the lower semicontinuity of the total variation, to (4.13) and to the hypotheses on the approximation r_o^h, for $t \in I$ we get

$$\begin{aligned}\mathrm{TV}\left(r(t)\right) \le {} & \lim_h \mathrm{TV}\left(r_h(t)\right) = \lim_h \left\|\nabla r_h(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^N)}\\ \le {} & \exp\left(\int_0^t \left\|\nabla u\,(\tau)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{N\times N})} \mathrm{d}\tau\right)\\ & \times\left(\lim_h \mathrm{TV}\,(r_o^h) + \lim_h \left\|r_o^h\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \int_0^t \left\|\nabla \operatorname{div} u(\tau)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^N)} \mathrm{d}\tau\right)\\ \le {} & \exp\left(\int_0^t \left\|\nabla u\,(\tau)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{N\times N})} \mathrm{d}\tau\right)\\ & \times\left(\mathcal{O}(1)\|r_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \mathrm{TV}\,(r_o) + \|r_o\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \int_0^t \left\|\nabla \operatorname{div} u(\tau)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^N)} \mathrm{d}\tau\right),\end{aligned}$$

concluding the proof of (4.10). The proof of the $\mathbf{L}^1$–Lipschitz continuity in time is done analogously, leading to (4.11).

Finally, (4.12) follows from (4.8), due to the linearity of (4.1). □

Remark 4.5. We underline that the total variation estimate just obtained differs from that presented in [13, Lemma 5.3], where the transport equation $\partial_t r + \operatorname{div}\left(r\, u(t,x)\right) = 0$ is studied not on a bounded domain Ω, but on all $\mathbb{R}^N$. Indeed, compare (4.10) and [13, Formula (5.12)]: it is immediate to see that, in the case of a divergence free vector field u, the $\mathbf{L}^\infty$–norm of the initial datum is still present in our case, while it is not in [13, Formula (5.12)]. This is actually due to the presence of the boundary.

Consider the following example to see the importance of the term $\|r_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}$ in (4.10). Let $\Omega = B(0,1) \subset \mathbb{R}^N$, $u(t,x) = -x$ and $r_o(x) = 2$ for every $x \in \Omega$. Then, the solution to (4.3) is $X(t;t_o,x_o) = x_o\, e^{t_o - t}$. Since $\operatorname{div} u = -N$, the solution to (4.1) is:

$$r(t,x) = \begin{cases} 2\, e^{N\,t} & \text{for } x \in B(0,e^{-t})\\ 0 & \text{elsewhere.}\end{cases}$$

Therefore, for every $t \in \mathbb{R}_+$, the total variation of $r(t)$ has contribution only from the *jump* between $2\,e^{N\,t}$ and 0, multiplied by the $(N-1)$ dimensional measure of the boundary $\partial B(0, e^{-t})$, that is

$$\mathrm{TV}\left(r(t)\right) = 2\,e^{N\,t}\,\frac{2\,\pi^{N/2}\,(e^{-t})^{N-1}}{\Gamma(N/2)} = 2\,\frac{\pi^{N/2}}{\Gamma(N/2)}\,e^{t},$$

Γ being the gamma function. Coherently, applying (4.10) we get

$$\mathrm{TV}\left(r(t)\right) \le e^{t}\,\mathcal{O}(1)\,\|r_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} = 2\,\mathcal{O}(1)\,e^{t}\,,$$

which confirms the necessity of the term $\|r_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}$ in the right hand side of (4.10).

We now provide a stability estimate of use below.

Lemma 4.6. *Let* **(Ω)** *hold. Let u and $\tilde u$ satisfy* **(u)**. *Assume $r_o \in (\mathbf{L}^\infty \cap \mathbf{BV})(\Omega;\mathbb{R})$. Call r and $\tilde r$ the solutions to* (4.1) *obtained with u and $\tilde u$, respectively. Then, for all $t \in I$,*

$$\begin{aligned}
&\left\|r(t) - \tilde r(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \\
&\le e^{\kappa(t)} \int_0^t \left\|(u-\tilde u)(s)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^N)} \mathrm{d}s \left[\mathcal{O}(1)\,\|r_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \mathrm{TV}\,(r_o) + \|r_o\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\,\kappa_1(t)\right] \\
&\quad + \|r_o\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \int_0^t \left\|\mathrm{div}\,(u-\tilde u)(s)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \mathrm{d}s\,,
\end{aligned} \tag{4.14}$$

where

$$\begin{aligned}
\kappa(t) &= \int_0^t \max\left\{\left\|\nabla u(s)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{N\times N})}, \left\|\nabla \tilde u(s)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{N\times N})}\right\} \mathrm{d}s\,,\\
\kappa_1(t) &= \int_0^t \max\left\{\left\|\nabla\,\mathrm{div}\,u(s)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{N})}, \left\|\nabla\,\mathrm{div}\,\tilde u(s)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{N})}\right\} \mathrm{d}s\,.
\end{aligned}$$

Proof. Regularise the initial datum r_o as in the proof of Lemma 4.3: for any $h \in \mathbb{N}\setminus\{0\}$ we have that $r_o^h \in \mathbf{C}^3(\Omega;\mathbb{R})$ converges to r_o in $\mathbf{L}^1(\Omega;\mathbb{R})$, $r_o^h(\xi) = 0$ for all $\xi \in \partial\Omega$ and (4.6)–(4.7) hold.

For $\vartheta \in [0,1]$, set

$$u_\vartheta(t,x) = \vartheta\,u(t,x) + (1-\vartheta)\,\tilde u(t,x).$$

Call r_ϑ^h the solution to (4.1) corresponding to the vector field u_ϑ above and to the initial datum r_o^h. Consider the map X_ϑ associated to u_ϑ, as in (4.3). We have that $r_\vartheta^h(t) \in \mathbf{C}^1(\Omega;\mathbb{R})$ for every $t \in I$ and it satisfies (4.4), that now reads as follow:

$$r_\vartheta^h(t,x) = \begin{cases} r_o^h\left(X_\vartheta(0;t,x)\right)\exp\left[-\int_0^t \mathrm{div}\,u_\vartheta\left(\tau, X_\vartheta(\tau;t,x)\right)\mathrm{d}\tau\right] & \text{if } x \in X_\vartheta(t;0,\Omega)\\ 0 & \text{elsewhere.}\end{cases} \tag{4.15}$$

Derive the analog of (4.3) with respect to ϑ and recall that $X_\vartheta(t;t,x) = x$ for all ϑ:

$$\begin{cases} \partial_t\partial_\vartheta X_\vartheta(\tau;t,x) = u(\tau, X_\vartheta(\tau;t,x)) - \tilde u(\tau, X_\vartheta(\tau;t,x)) + \nabla u_\vartheta(\tau, X_\vartheta(\tau;t,x))\,\partial_\vartheta X_\vartheta(\tau;t,x)\\ \partial_\vartheta X_\vartheta(t;t,x) = 0\,.\end{cases}$$

The solution to this problem is given by

$$\begin{aligned}
\partial_\vartheta X_\vartheta(\tau;t,x) &= \int_t^\tau \exp\left(\int_s^\tau \nabla u_\vartheta(\sigma, X_\vartheta(\sigma;t,x))\,\mathrm{d}\sigma\right)\Big(u\left(s, X_\vartheta(s;t,x)\right) - \tilde u\left(s, X_\vartheta(s;t,x)\right)\Big)\mathrm{d}s\\
&= \int_\tau^t \exp\left(\int_\tau^s -\nabla u_\vartheta(\sigma, X_\vartheta(\sigma;t,x))\,\mathrm{d}\sigma\right)(\tilde u - u)\big(s, X_\vartheta(s;t,x)\big)\,\mathrm{d}s\,.
\end{aligned} \tag{4.16}$$

Derive now the non zero expression in the right hand side of (4.15) with respect to ϑ:

$$
\begin{aligned}
& \partial_\vartheta r^h_\vartheta(t,x) \\
= \ & \exp\left(\int_0^t -\operatorname{div} u_\vartheta\left(\tau, X_\vartheta(\tau;t,x)\right) \mathrm{d}\tau\right) \\
& \times \left\{ \nabla r^h_o\left(X_\vartheta(0;t,x)\right) \partial_\vartheta X_\vartheta(0;t,x) + r^h_o\left(X_\vartheta(0;t,x)\right) \int_0^t \operatorname{div}(\tilde u - u)\bigl(\tau, X_\vartheta(\tau;t,x)\bigr) \mathrm{d}\tau \right. \\
& \qquad \left. - r^h_o\left(X_\vartheta(0;t,x)\right) \int_0^t \nabla \operatorname{div} u_\vartheta\left(\tau, X_\vartheta(\tau;t,x)\right) \cdot \partial_\vartheta X_\vartheta(\tau;t,x) \,\mathrm{d}\tau \right\} \\
= \ & \exp\left(\int_0^t -\operatorname{div} u_\vartheta\left(\tau, X_\vartheta(\tau;t,x)\right) \mathrm{d}\tau\right) \\
& \times \left\{ \nabla r^h_o\left(X_\vartheta(0;t,x)\right) \int_0^t \exp\left(\int_0^s -\nabla u_\vartheta(\sigma, X_\vartheta(\sigma;t,x))\,\mathrm{d}\sigma\right) (\tilde u - u)\bigl(s, X_\vartheta(s;t,x)\bigr) \mathrm{d}s \right. \\
& + r^h_o\left(X_\vartheta(0;t,x)\right) \int_0^t \operatorname{div}(\tilde u - u)\bigl(\tau, X_\vartheta(\tau;t,x)\bigr) \mathrm{d}\tau \\
& - r^h_o\left(X_\vartheta(0;t,x)\right) \int_0^t \nabla \operatorname{div} u_\vartheta\left(\tau, X_\vartheta(\tau;t,x)\right) \\
& \quad \left. \times \left[\int_\tau^t \exp\left(\int_\tau^s -\nabla u_\vartheta(\sigma, X_\vartheta(\sigma;t,x))\,\mathrm{d}\sigma\right) (\tilde u - u)\bigl(s, X_\vartheta(s;t,x)\bigr) \mathrm{d}s\right] \mathrm{d}\tau \right\},
\end{aligned}
$$

where we used (4.16). Call r^h and $\tilde r^h$ the solutions to (4.1) corresponding to velocities u and $\tilde u$ respectively, and initial datum r^h_o: in other words, $r^h = r^h_{\vartheta=1}$, while $\tilde r^h = r^h_{\vartheta=0}$. Compute

$$
\left\| r^h(t) - \tilde r^h(t) \right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \le \int_\Omega \left| \int_0^1 \partial_\vartheta r^h_\vartheta(t,x)\,\mathrm{d}\vartheta \right| \mathrm{d}x \le \int_0^1 \int_{X_\vartheta(t;0,\Omega)} \left| \partial_\vartheta r^h_\vartheta(t,x) \right| \mathrm{d}x\,\mathrm{d}\vartheta\,. \tag{4.17}
$$

In particular, introduce the change of variable for X_ϑ analogous to that presented at the beginning of the proof of Lemma 4.4, set $Y = X_\vartheta\left(0;t,X_\vartheta(t;0,\Omega)\right)$ and compute

$$
\begin{aligned}
& \int_{X_\vartheta(t;0,\Omega)} \left| \partial_\vartheta r^h_\vartheta(t,x) \right| \mathrm{d}x \\
\le \ & \int_Y \left| \nabla r^h_o(y) \int_0^t \exp\left(\int_0^s -\nabla u_\vartheta\left(\sigma, X_\vartheta(\sigma;0,y)\right) \mathrm{d}\sigma\right) (\tilde u - u)\bigl(s, X_\vartheta(s;0,y)\bigr) \mathrm{d}s \right| \mathrm{d}y \\
& + \int_Y \left| r^h_o(y) \int_0^t \operatorname{div}(\tilde u - u)\bigl(\tau, X_\vartheta(\tau;0,y)\bigr) \mathrm{d}\tau \right| \mathrm{d}y \\
& + \int_Y \left| r^h_o(y) \int_0^t \nabla \operatorname{div} u_\vartheta\left(\tau, X_\vartheta(\tau;0,y)\right) \right. \\
& \qquad \left. \times \int_\tau^t \exp\left(\int_\tau^s -\nabla u_\vartheta\left(\sigma, X_\vartheta(\sigma;0,y)\right) \mathrm{d}\sigma\right) (\tilde u - u)\bigl(s, X_\vartheta(s;0,y)\bigr) \mathrm{d}s\,\mathrm{d}\tau \right| \mathrm{d}y \\
\le \ & \left(\int_\Omega \left| \nabla r^h_o(y) \right| \mathrm{d}y\right) \exp\left(\int_0^t \left\| \nabla u_\vartheta(s) \right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{N\times N})} \mathrm{d}s\right) \int_0^t \left\| (u - \tilde u)(s) \right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^N)} \mathrm{d}s \\
& + \left\| r^h_o \right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \int_0^t \left\| \operatorname{div}(u - \tilde u)(s) \right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \mathrm{d}s \\
& + \left\| r^h_o \right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \int_0^t \left\| \nabla \operatorname{div} u_\vartheta(s) \right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^N)} \mathrm{d}s
\end{aligned}
$$

$$\times \exp\left(\int_0^t \left\|\nabla u_\vartheta(s)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{N\times N})} \mathrm{d}s\right) \int_0^t \left\|(u-\tilde u)(s)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^N)} \mathrm{d}s\,.$$

Therefore, inserting the latter result above in (4.17) yields

$$\begin{aligned}
&\left\|r^h(t)-\tilde r^h(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} &(4.18)\\
&\le \exp\left(\int_0^t \max\left\{\left\|\nabla u(s)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{N\times N})}, \left\|\nabla \tilde u(s)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{N\times N})}\right\} \mathrm{d}s\right) \int_0^t \left\|(u-\tilde u)(s)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^N)} \mathrm{d}s\\
&\quad\times\left[\int_\Omega \left|\nabla r_o^h(y)\right| \mathrm{d}y + \left\|r_o^h\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \int_0^t \max\left\{\left\|\nabla \operatorname{div} u(s)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^N)}, \left\|\nabla \operatorname{div} \tilde u(s)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^N)}\right\} \mathrm{d}s\right]\\
&\quad+\left\|r_o^h\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \int_0^t \left\|\operatorname{div}(u-\tilde u)(s)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \mathrm{d}s\,. &(4.19)
\end{aligned}$$

We now let h tend to $+\infty$. We know that r_o^h converges to r_o in $\mathbf{L}^1(\Omega;\mathbb{R})$, so that r_ϑ^h, solution to (4.1) with velocity u_ϑ and initial datum r_o^h, converges to a function r_ϑ in $\mathbf{L}^1$ which is solution to (4.1) with velocity u_ϑ and initial datum r_o. Call $r = r_{\vartheta=1}$ and $\tilde r = r_{\vartheta=0}$: they are solutions to (4.1) with velocities u and $\tilde u$ respectively, and initial datum r_o. It is clear that $r^h \to r$ and $\tilde r^h \to \tilde r$ in $\mathbf{L}^1$. Therefore, the inequality (4.18)–(4.19) in the limit $h \to +\infty$ reads

$$\begin{aligned}
&\left\|r(t)-\tilde r(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\\
&\le \exp\left(\int_0^t \max\left\{\left\|\nabla u(s)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{N\times N})}, \left\|\nabla \tilde u(s)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{N\times N})}\right\} \mathrm{d}s\right) \int_0^t \left\|(u-\tilde u)(s)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^N)} \mathrm{d}s\\
&\quad\times\Bigg[\mathcal{O}(1)\left\|r_o\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \mathrm{TV}(r_o)\\
&\qquad+\left\|r_o\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \int_0^t \max\left\{\left\|\nabla \operatorname{div} u(s)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^N)}, \left\|\nabla \operatorname{div} \tilde u(s)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^N)}\right\} \mathrm{d}s\Bigg]\\
&\quad+\left\|r_o\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \int_0^t \left\|\operatorname{div}(u-\tilde u)(s)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \mathrm{d}s\,,
\end{aligned}$$

where we used the fact that $\int_\Omega \left|\nabla r_o^h(y)\right| \mathrm{d}y = \mathrm{TV}(r_o^h)$ and (4.7). □

Proof of Theorem 2.2. The proof relies on a fixed point argument and consists of several steps. Fix $R = \max\left\{\left\|\rho_o\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}, \left\|\rho_o\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)}, \mathrm{TV}(\rho_o)\right\}$. Given a map $\mathcal{F}(t) \in \mathbf{C}^0(I;\mathbb{R}_+)$, whose precise choice is given in the sequel, the following functional space is of use below:

$$\mathcal{X}_R = \left\{ r \in \mathbf{C}^0(I;\mathbf{L}^1(\Omega;\mathbb{R}^n))\colon \begin{array}{l} \left\|r\right\|_{\mathbf{L}^\infty(I;\mathbf{L}^1(\Omega;\mathbb{R}^n))} \le R \text{ and}\\ \left\|r(t)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)} < +\infty \text{ for all } t \in I\\ \mathrm{TV}\left(r(t)\right) \le \mathcal{F}(t) \text{ for all } t \in I \end{array} \right\} \tag{4.20}$$

with the distance $d(\rho_1,\rho_2) = \left\|\rho_1-\rho_2\right\|_{\mathbf{L}^\infty(I;\mathbf{L}^1(\Omega;\mathbb{R}^n))}$, so that $\mathcal{X}_R$ is a complete metric space.

Throughout, we denote by C a positive constant that depends on the assumptions **(Ω)**, **(V)**, **(J)**, on R and on n. The constant C does not depend on time. For the sake of simplicity, introduce the notation $\Sigma_t = [0,t] \times \Omega \times \mathbb{R}^m$.

Reduction to a Fixed Point Problem. Define the map

$$\begin{array}{rccc} \mathcal{T} : & \mathcal{X}_R & \to & \mathcal{X}_R\\ & r & \to & \rho \end{array} \tag{4.21}$$

where $\rho \equiv (\rho^1, \ldots, \rho^n)$ solves

$$\left\{\begin{array}{lll}\partial_t \rho^i + \operatorname{div}\left[\rho^i\, V^i\left(t, x, \left(\mathcal{J}^i r(t)\right)(x)\right)\right] = 0 & (t,x) \in I \times \Omega & i = 1, \ldots, n \\ \rho(t,\xi) = 0 & (t,\xi) \in I \times \partial\Omega & \\ \rho(0,x) = \rho_o(x) & \quad x \in \Omega. & \end{array}\right. \tag{4.22}$$

A map $\rho \in \mathcal{X}_R$ solves (1.1) in the sense of Definition 2.1 if and only if ρ is a fixed point for $\mathcal{T}$.

$\mathcal{T}$ is Well Defined. Given $r \in \mathcal{X}_R$, by **(V)** and **(J)**, for $i = 1, \ldots, n$ each map

$$u^i(t,x) = V^i\left(t, x, \left(\mathcal{J}^i r(t)\right)(x)\right) \tag{4.23}$$

satisfies **(u)**. The solution ρ to (4.22) is well defined, unique and belongs to $\mathbf{C}^0(I; \mathbf{L}^1(\Omega; \mathbb{R}^n))$. With the notation introduced above, by (4.8) in Lemma 4.4, for all $t \in I$,

$$\left\|\rho(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} \le \left\|\rho_o\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} \tag{4.24}$$

and, by **(V)**, **(J)** and (4.9),

$$\begin{aligned}\left\|\rho^i(t)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} &\le \left\|\rho^i_o\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \exp\left[t\left\|\operatorname{div} V^i\right\|_{\mathbf{L}^\infty(\Sigma_t;\mathbb{R})} + tK\left\|\nabla_w V^i\right\|_{\mathbf{L}^\infty(\Sigma_t;\mathbb{R}^{N\times m})}\left\|r(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\right] \\ &\le \left\|\rho^i_o\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \exp\left(t\,\mathcal{V}\,(1 + K\,R)\right) \\ &\le \left\|\rho^i_o\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} e^{C\,t} \qquad\qquad \text{for } i = 1, \ldots, n, \text{ so that}\end{aligned}$$

$$\left\|\rho(t)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)} \le \left\|\rho_o\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)} e^{C\,t}\,. \tag{4.25}$$

Applying (4.10) in Lemma 4.4, with the help of **(V)** and **(J)**, for all $t \in I$ and all $i = 1, \ldots, n$,

$$\operatorname{TV}\left(\rho^i(t)\right) \le \exp\left(\int_0^t \left\|\nabla u^i(\tau)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{N\times N})} \mathrm{d}\tau\right) \tag{4.26}$$

$$\begin{aligned}&\times\left(\mathcal{O}(1)\left\|\rho^i_o\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \operatorname{TV}(\rho^i_o) + \left\|\rho^i_o\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\int_0^t \left\|\nabla \operatorname{div} u^i(\tau)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^N)} \mathrm{d}\tau\right) \\ &\le \exp\left(t\left\|\nabla V^i\right\|_{\mathbf{L}^\infty(\Sigma_t;\mathbb{R}^{N\times N})} + t\,K\left\|\nabla_w V^i\right\|_{\mathbf{L}^\infty(\Sigma_t;\mathbb{R}^{N\times m})}\left\|r(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\right) \\ &\quad\times\left[\mathcal{O}(1)\left\|\rho^i_o\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \operatorname{TV}(\rho^i_o) + t\left\|\rho^i_o\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\left(\left\|\nabla_x \operatorname{div} V^i\right\|_{\mathbf{L}^\infty(\Sigma_t;\mathbb{R}^N)}\right.\right. \\ &\qquad + K\left(\left\|\nabla_w \operatorname{div} V^i\right\|_{\mathbf{L}^\infty(\Sigma_t;\mathbb{R}^m)} + \left\|\nabla_x \nabla_w V^i\right\|_{\mathbf{L}^\infty(\Sigma_t;\mathbb{R}^{N\times m\times N})}\right)\left\|r(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} \\ &\qquad + K^2\left\|\nabla^2_{ww} V^i\right\|_{\mathbf{L}^\infty(\Sigma_t;\mathbb{R}^{N\times m\times m})}\left\|r(t)\right\|^2_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} \\ &\qquad \left.\left. + \left\|\nabla_w V^i\right\|_{\mathbf{L}^\infty(\Sigma_t;\mathbb{R}^{N\times m})}\mathcal{K}\left(\left\|r(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\right)\left\|r(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\right)\right]\end{aligned}$$

$$\le \left(C\,t + C\left\|\rho^i_o\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \operatorname{TV}(\rho^i_o)\right) e^{Ct}\,, \tag{4.27}$$

so that

$$\operatorname{TV}\left(\rho(t)\right) \le \left(C\,t + C\left\|\rho_o\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)} + \operatorname{TV}(\rho_o)\right) e^{Ct}\,. \tag{4.28}$$

The map $\mathcal{T}$ is thus well defined, setting in (4.20)

$$\mathcal{F}(t) = \left(C\,t + C\,\|\rho_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)} + \mathrm{TV}\,(\rho_o) \right) e^{Ct}\,. \tag{4.29}$$

$\mathcal{T}$ is a Contraction. For any $r_1, r_2 \in \mathcal{X}_R$, denote for $j = 1, 2$, $\rho_j = \mathcal{T}(r_j)$ and, correspondingly, u^i_j as in (4.23) for $i = 1, \ldots, n$. Compute, thanks to **(V)** and **(J)**,

$$\begin{aligned}
\left\| \nabla u^i_j(t) \right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{N\times N})} &\le \left\| \nabla_x V^i \right\|_{\mathbf{L}^\infty(\Sigma_t;\mathbb{R}^{N\times N})} + \left\| \nabla_w V^i \right\|_{\mathbf{L}^\infty(\Sigma_t;\mathbb{R}^{N\times m})} \left\| \nabla_x \mathcal{J}^i r_j(t) \right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{m\times N})} \\
&\le \mathcal{V} \left(1 + K \left\| r_j(t) \right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} \right) \\
&\le \mathcal{V}\,(1 + K\,R) \\
&\le C
\end{aligned}$$

and

$$\begin{aligned}
&\left\| \nabla\,\mathrm{div}\, u^i_j(t) \right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^N)} \\
\le\ & \left\| \nabla_x\,\mathrm{div}\, V^i \right\|_{\mathbf{L}^\infty(\Sigma_t;\mathbb{R}^N)} \\
&+ \left(\left\| \nabla_w\,\mathrm{div}\, V^i \right\|_{\mathbf{L}^\infty(\Sigma_t;\mathbb{R}^m)} + \left\| \nabla_x \nabla_w V^i \right\|_{\mathbf{L}^\infty(\Sigma_t;\mathbb{R}^{N\times m\times N})} \right) \left\| \nabla_x \mathcal{J}^i r_j(t) \right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{m\times N})} \\
&+ \left\| \nabla^2_w V^i \right\|_{\mathbf{L}^\infty(\Sigma_t;\mathbb{R}^{N\times m\times m})} \left\| \nabla_x \mathcal{J}^i r_j(t) \right\|^2_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{m\times N})} \\
&+ \left\| \nabla_w V^i \right\|_{\mathbf{L}^\infty(\Sigma_t;\mathbb{R}^{N\times m})} \left\| \nabla^2_x \mathcal{J}^i r_j(t) \right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{m\times N\times N})} \\
\le\ & \mathcal{V} \left(1 + K \left\| r_j(t) \right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} + K^2 \left\| r_j(t) \right\|^2_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} + \mathcal{K}\Big(\left\| r_j(t) \right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} \Big) \left\| r_j(t) \right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} \right) \\
\le\ & \mathcal{V} \left(1 + K\,R + K^2\,R^2 + \mathcal{K}(R)\ R \right) \\
\le\ & C\,.
\end{aligned}$$

Furthermore, still using assumption **(J)**, we have that, for all $t \in I$,

$$\begin{aligned}
\left\| (u^i_2 - u^i_1)(t) \right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^N)} &\le \left\| \nabla_w V^i \right\|_{\mathbf{L}^\infty(\Sigma_t;\mathbb{R}^{N\times m})} \left\| \mathcal{J}^i r_2(t) - \mathcal{J}^i r_1(t) \right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^m)} \\
&\le \mathcal{V}\,K \left\| r_2(t) - r_1(t) \right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} \\
&\le C \left\| r_2(t) - r_1(t) \right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\,. \\
\left\| \mathrm{div}\,(u^i_2 - u^i_1)(t) \right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} &\le \left\| \nabla_w\,\mathrm{div}\, V^i \right\|_{\mathbf{L}^\infty(\Sigma_t;\mathbb{R}^m)} \left\| \mathcal{J}^i r_2(t) - \mathcal{J}^i r_1(t) \right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^m)} \\
&\quad + \left\| \nabla_w V^i \right\|_{\mathbf{L}^\infty(\Sigma_t;\mathbb{R}^{N\times m})} \left\| \nabla_x \mathcal{J}^i r_2(t) - \nabla_x \mathcal{J}^i r_1(t) \right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{m\times N})} \\
&\le \mathcal{V} \left(K + \mathcal{K} \left(\left\| r_1(t) \right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} \right) \right) \left\| r_2(t) - r_1(t) \right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} \\
&\le C \left\| r_2(t) - r_1(t) \right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\,.
\end{aligned}$$

Therefore, for all $t \in I$, by Lemma 4.6, with obvious notation we have

$$\begin{aligned}
&\left\| \rho^i_2(t) - \rho^i_1(t) \right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \\
\le\ & e^{\kappa(t)} \int_0^t \left\| (u^i_2 - u^i_1)(s) \right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^N)} \mathrm{d}s \left[\mathcal{O}(1) \left\| \rho^i_o \right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \mathrm{TV}\,(\rho^i_o) + \kappa_1(t) \left\| \rho^i_o \right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \right]
\end{aligned}$$

$$
\begin{aligned}
&+ \left\|\rho_o^i\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \int_0^t \left\|\operatorname{div}(u_2^i - u_1^i)(s)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \mathrm{d}s \\
\le{}& C\,t \left[e^{C\,t}\left(\mathcal{O}(1)\left\|\rho_o\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)} + \mathrm{TV}\,(\rho_o) + C\,t\right) + C\right] \left\|r_2 - r_1\right\|_{\mathbf{L}^\infty([0,t];\mathbf{L}^1(\Omega;\mathbb{R}^n))} \\
\le{}& C\,t \left[e^{C\,t}\left(C\left\|\rho_o\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)} + \mathrm{TV}\,(\rho_o) + C\,t\right) + C\right] \left\|r_2 - r_1\right\|_{\mathbf{L}^\infty([0,t];\mathbf{L}^1(\Omega;\mathbb{R}^n))},
\end{aligned}
$$

We obtain that $\mathcal{T}$ is a contraction when restricted to the time interval $[0, T_1]$, with T_1 such that

$$
C\,T_1 \left[e^{C\,T_1}\left(C\left\|\rho_o\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)} + \mathrm{TV}\,(\rho_o) + C\,T_1\right) + C\right] = \frac{1}{2}\,. \tag{4.30}
$$

Existence of a solution on $[0, T_1]$. By the steps above, there exists a fixed point $\rho_1 \in \mathcal{X}_R$ for the map $\mathcal{T}$ defined in (4.21), restricted to functions defined on the time interval $[0, T_1]$. By construction, ρ_1 solves (1.1) on the time interval $[0, T_1]$.

Existence of a solution on I. We consider two cases: $I = \mathbb{R}_+$ and $I = [0, T]$, for a fixed positive T. If, in the second case, $T_1 \ge \sup I$, the statement obviously holds. Otherwise, if $T_1 < \sup I$, we extend ρ_1 to I by iterating the procedure above.

Assume that the solution exists up to the time $T_{k-1} < \sup I$. Thanks to the bounds (4.25) and (4.27), define recursively T_k so that

$$
\begin{aligned}
C\,(T_k - T_{k-1}) \Big[\Big(2\,C\left\|\rho_o\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)} + \mathrm{TV}\,(\rho_o) + C\,T_{k-1}\Big)\,e^{C\,T_k}& \\
+ C\,(T_k - T_{k-1})\,e^{C\,(T_k - T_{k-1})} + C\Big] = \frac{1}{2}.&
\end{aligned} \tag{4.31}
$$

Indeed, the above procedure ensures that there exists a fixed point for the map $\mathcal{T}$ defined in (4.21), restricted to functions defined on the time interval $[T_{k-1}, T_k]$. If, in the case of the time interval $I = [0, T]$, $T_k \ge \sup I$, the statement is proved. Otherwise, if we assume that the sequence (T_k) remains less than $\sup I$, it is in particular bounded. Hence, the left hand side of the relation above tends to 0, while the right hand side is $1/2 > 0$. Therefore, the sequence (T_k) is unbounded, ensuring that, for k large, T_k is greater than $\sup I$, thus the solution to (1.1) is defined on all I.

Bounds on the solution. The $\mathbf{L}^1$–bound follows immediately by the construction of the solution. By (4.25) we have

$$
\begin{aligned}
&\left\|\rho^i(t)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \le \left\|\rho_o^i\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \exp\left(t\,\mathcal{V}\left(1 + K\left\|\rho(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\right)\right) \qquad \text{whence} \\
&\left\|\rho(t)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)} \le \left\|\rho_o\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)} \exp\left(t\,\mathcal{V}\left(1 + K\left\|\rho_o\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\right)\right).
\end{aligned}
$$

Moreover, by (4.26)–(4.27)

$$
\begin{aligned}
\mathrm{TV}\Big(\rho^i(t)\Big) \le{}& \exp\left(t\,\mathcal{V}\left(1 + K\left\|\rho(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\right)\right) \\
&\times\Bigg(\mathcal{O}(1)\left\|\rho_o^i\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \mathrm{TV}\left(\rho_o^i\right) + t\left\|\rho_o^i\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\mathcal{V} \\
&\times\Big(1 + K\left\|\rho(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} + K^2\left\|\rho(t)\right\|^2_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} + \mathcal{K}\Big(\left\|\rho(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\Big)\left\|\rho(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\Big)\Bigg). \\
\mathrm{TV}\Big(\rho(t)\Big) \le{}& \exp\left(t\,\mathcal{V}\left(1 + K\left\|\rho_o\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\right)\right)
\end{aligned}
$$

$$
\times \left(\mathcal{O}(1) n \left\|\rho_o\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)} + \mathrm{TV}\,(\rho_o) + n\, t \left\|\rho_o\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} \mathcal{V} \right.
$$

$$
\left. \times \left(1 + K \left\|\rho_o\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} + K^2 \|\rho_o\|^2_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} + \mathcal{K}\Big(\left\|\rho_o\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\Big) \left\|\rho_o\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\right)\right),
$$

concluding the proof of (2).

Lipschitz dependence on time. Apply (4.11) in Lemma 4.4 and the total variation estimate obtained in the previous step: for any $t,\, s \in I$

$$
\left\|\rho(t) - \rho(s)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} \leq \mathrm{TV}\left(\rho\left(\max\{t,s\}\right)\right)\, |t-s|.
$$

Lipschitz dependence on the initial datum. Assume that $I = [0,t]$, so that $\lim\limits_{k\to+\infty} T_k = t$, where T_k is defined recursively through (4.31), which can be rewritten as follows:

$$
C\,(T_k - T_{k-1}) \left[\left((2\,C+1)\,R + C\,T_{k-1}\right) e^{C\,T_k} + C\,(T_k - T_{k-1})\, e^{C\,(T_k - T_{k-1})} + C\right] = \frac{1}{2}, \qquad (4.32)
$$

the constant C depending on the assumptions **(Ω)**, **(V)**, **(J)** and on R, which is now defined as

$$
R = \max\left\{\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)},\, \|\tilde\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)},\, \|\rho_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)},\, \|\tilde\rho_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)},\, \mathrm{TV}\,(\rho_o),\, \mathrm{TV}\,(\tilde\rho_o)\right\}.
$$

To make evident the dependence of $\mathcal{T}$ on the initial datum, introduce the space

$$
\mathcal{Y}_R = \left\{\rho_o \in (\mathbf{L}^\infty \cap \mathbf{BV})(\Omega;\mathbb{R}^n) \colon \|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} \leq R,\, \|\rho_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)} \leq R,\, \mathrm{TV}\,(\rho_o) \leq R\right\}
$$

and slightly modify the map $\mathcal{T}$ to

$$
\begin{array}{rccl}
\mathcal{T} & : & \mathcal{X}_R \times \mathcal{Y}_R & \to & \mathcal{X}_R \\
& & r,\, \rho_o & \to & \rho
\end{array}
$$

where ρ solves (4.22). The map $\mathcal{T}$ is a contraction in $r \in \mathcal{X}_R$, Lipschitz continuous in $\rho_o \in \mathcal{Y}_R$, when restricted to functions defined on each time interval $[T_k, T_{k+1}]$. In particular,

$$
\begin{aligned}
& \left\|\mathcal{T}(r, \rho(T_k)) - \mathcal{T}(\tilde r, \tilde\rho(T_k))\right\|_{\mathbf{L}^\infty([T_k,T_{k+1}];\mathbf{L}^1(\Omega;\mathbb{R}^n))} \\
\leq\ & \frac{1}{2}\, \|r - \tilde r\|_{\mathbf{L}^\infty([T_k,T_{k+1}];\mathbf{L}^1(\Omega;\mathbb{R}^n))} + \left\|\rho(T_k) - \tilde\rho(T_k)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}
\end{aligned}
$$

by (4.32) and (4.12). Hence, $\left\|\rho(T_k) - \tilde\rho(T_k)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} \leq 2\left\|\rho(T_{k-1}) - \tilde\rho(T_{k-1})\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}$, which recursively yields $\left\|\rho(T_k) - \tilde\rho(T_k)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} \leq 2^k \left\|\rho_o - \tilde\rho_o\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}$. The term in square brackets in the left hand side of (4.32) is uniformly bounded in k by a positive constant, say, A_t. Therefore, $T_k \geq \dfrac{1}{2\,A_t\,C} + T_{k-1}$ which recursively yields $T_k \geq k/(2\,A_t\,C)$ and $k \leq 2\,A_t\,C\,T_k < 2\,A_t\,C\,t$, so that

$$
\left\|\rho(t) - \tilde\rho(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} = \lim_{k\to+\infty} \left\|\rho(T_k) - \tilde\rho(T_k)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} \leq 2^{2\,A_t\,C\,t} \left\|\rho_o - \tilde\rho_o\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}
$$

completing the proof of (4).

Stability estimate. We aim to apply (4.14) in Lemma 4.6. Exploit the definition $u^i(t,x) = V^i\left(t, x, \left(\mathcal{J}^i\rho(t)\right)(x)\right)$ and compute, thanks to **(V)** and **(J)**:

$$
\begin{aligned}
\left\|\nabla u^i(t)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{N\times N})} &\leq \left\|\nabla_x V^i\right\|_{\mathbf{L}^\infty(\Sigma_t;\mathbb{R}^{N\times N})} + \left\|\nabla_w V^i\right\|_{\mathbf{L}^\infty(\Sigma_t;\mathbb{R}^{N\times m})}\left\|\nabla_x \mathcal{J}^i\rho(t)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{m\times N})}\\
&\leq \mathcal{V}\left(1 + K\left\|\rho(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\right)\\
&\leq \mathcal{V}\left(1 + K\left\|\rho_o\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\|\nabla \operatorname{div} u^i(t)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^N)}\\
\leq\; & \left\|\nabla_x \operatorname{div} V^i\right\|_{\mathbf{L}^\infty(\Sigma_t;\mathbb{R}^N)}\\
&+\left(\left\|\nabla_w \operatorname{div} V^i\right\|_{\mathbf{L}^\infty(\Sigma_t;\mathbb{R}^m)} + \left\|\nabla_x\nabla_w V^i\right\|_{\mathbf{L}^\infty(\Sigma_t;\mathbb{R}^{N\times m\times N})}\right)\left\|\nabla_x\mathcal{J}^i\rho(t)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{m\times N})}\\
&+\left\|\nabla_w^2 V^i\right\|_{\mathbf{L}^\infty(\Sigma_t;\mathbb{R}^{N\times m\times m})}\left\|\nabla_x\mathcal{J}^i\rho(t)\right\|^2_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{m\times N})}\\
&+\left\|\nabla_w V^i\right\|_{\mathbf{L}^\infty(\Sigma_t;\mathbb{R}^{N\times m})}\left\|\nabla_x^2\mathcal{J}^i\rho(t)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{m\times N\times N})}\\
\leq\; & \mathcal{V}\left(1 + K\left\|\rho(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} + K^2\left\|\rho(t)\right\|^2_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} + \mathcal{K}\Bigl(\left\|\rho(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\Bigr)\left\|\rho(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\right)\\
\leq\; & \mathcal{V}\left(1 + \left\|\rho(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\left(K + K^2\left\|\rho_o\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} + \mathcal{K}\Bigl(\left\|\rho_o\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\Bigr)\right)\right),
\end{aligned}
$$

and the same estimates hold for each $\tilde u^i$, defined by $\tilde u^i(t,x) = \tilde V^i\left(t, x, \left(\mathcal{J}^i\tilde\rho(t)\right)(x)\right)$.

Moreover, still by **(V)** and **(J)**,

$$
\begin{aligned}
\left\|(u^i - \tilde u^i)(t)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^N)} &= \operatorname*{ess\,sup}_{x\in\Omega}\left|V^i\left(t,x,\left(\mathcal{J}^i\rho(t)\right)(x)\right) - \tilde V^i\left(t,x,\left(\mathcal{J}^i\tilde\rho(t)\right)(x)\right)\right|\\
&\leq \operatorname*{ess\,sup}_{x\in\Omega}\left|V^i\left(t,x,\left(\mathcal{J}^i\rho(t)\right)(x)\right) - V^i\left(t,x,\left(\mathcal{J}^i\tilde\rho(t)\right)(x)\right)\right|\\
&\quad + \operatorname*{ess\,sup}_{x\in\Omega}\left|V^i\left(t,x,\left(\mathcal{J}^i\tilde\rho(t)\right)(x)\right) - \tilde V^i\left(t,x,\left(\mathcal{J}^i\tilde\rho(t)\right)(x)\right)\right|\\
&\leq \left\|\nabla_w V^i\right\|_{\mathbf{L}^\infty(\Sigma_t;\mathbb{R}^{N\times m})}\left\|\mathcal{J}^i\rho(t) - \mathcal{J}^i\tilde\rho(t)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^m)}\\
&\quad + \left\|(V^i - \tilde V^i)(t)\right\|_{\mathbf{L}^\infty(\Omega\times B(0,K\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)});\mathbb{R}^N)}\\
&\leq \mathcal{V}K\left\|(\rho-\tilde\rho)(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} + \left\|(V^i - \tilde V^i)(t)\right\|_{\mathbf{L}^\infty(\Omega\times B(0,K\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)});\mathbb{R}^N)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\|\operatorname{div}(u^i - \tilde u^i)(t)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}\\
\leq\; & \operatorname*{ess\,sup}_{x\in\Omega}\left|\operatorname{div}\left(V^i\left(t,x,\left(\mathcal{J}^i\rho(t)\right)(x)\right) - \tilde V^i\left(t,x,\left(\mathcal{J}^i\tilde\rho(t)\right)(x)\right)\right)\right|\\
&+\operatorname*{ess\,sup}_{x\in\Omega}\left|\nabla_w V^i\left(t,x,\left(\mathcal{J}^i\rho(t)\right)(x)\right)\cdot\nabla\left(\mathcal{J}^i\rho(t)\right)(x)\right.
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad -\nabla_w \tilde V^i\left(t,x,\left(\mathcal{J}^i\tilde\rho(t)\right)(x)\right)\cdot\nabla\left(\mathcal{J}^i\tilde\rho(t)\right)(x)\Big|\\
&\le \left\|\nabla_w \operatorname{div} V^i\right\|_{\mathbf{L}^\infty(\Sigma_t;\mathbb{R}^m)}\left\|\mathcal{J}^i\rho(t)-\mathcal{J}^i\tilde\rho(t)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^m)}\\
&\quad+\left\|\operatorname{div}\left(V^i-\tilde V^i\right)(t)\right\|_{\mathbf{L}^\infty(\Omega\times B(0,K\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)});\mathbb{R})}\\
&\quad+\left\|\nabla_w V^i\right\|_{\mathbf{L}^\infty(\Sigma_t;\mathbb{R}^{N\times m})}\left\|\nabla\mathcal{J}^i\rho(t)-\nabla\mathcal{J}^i\tilde\rho(t)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{m\times N})}\\
&\quad+\left\|\nabla_w V^i(t)-\nabla_w\tilde V^i(t)\right\|_{\mathbf{L}^\infty(\Omega\times B(0,K\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)});\mathbb{R}^{N\times m})}\left\|\nabla\mathcal{J}^i\tilde\rho(t)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{m\times N})}\\
&\le \mathcal{V}\,K\left\|(\rho-\tilde\rho)(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}+\left\|\operatorname{div}\left(V^i-\tilde V^i\right)(t)\right\|_{\mathbf{L}^\infty(\Omega\times B(0,K\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)});\mathbb{R})}\\
&\quad+\mathcal{V}\,\mathcal{K}\Big(\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\Big)\left\|(\rho-\tilde\rho)(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\\
&\quad+K\|\rho_o\|_{\mathbf{L}^1(\Omega,\mathbb{R}^n)}\left\|\nabla_w\left(V^i-\tilde V^i\right)(t)\right\|_{\mathbf{L}^\infty(\Omega\times B(0,K\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)});\mathbb{R}^{N\times m})}.
\end{aligned}
$$

Therefore, for all $t\in I$, by (4.14) in Lemma 4.6, we have

$$
\begin{aligned}
&\left\|\rho^i(t)-\tilde\rho^i(t)\right\|_{\mathbf{L}^1(\Omega,\mathbb{R})}\\
&\le\ \exp\left(t\,\mathcal{V}\left(1+K\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\right)\right)\Bigg[\mathcal{O}(1)\|\rho_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)}+\mathrm{TV}(\rho_o)\\
&\qquad+t\,\mathcal{V}\,\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\left(1+\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\left(K+\mathcal{K}\Big(\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\Big)+K^2\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\right)\right)\Bigg]\\
&\times\left(\mathcal{V}\,K\int_0^t\|\rho(s)-\tilde\rho(s)\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\,\mathrm{d}s+\int_0^t\left\|(V^i-\tilde V^i)(s)\right\|_{\mathbf{L}^\infty(\Omega\times B(0,K\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)});\mathbb{R}^N)}\mathrm{d}s\right)\\
&+\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\,\mathcal{V}\left(K+\mathcal{K}\Big(\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\Big)\right)\int_0^t\|\rho(s)-\tilde\rho(s)\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\,\mathrm{d}s\\
&\quad+\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\int_0^t\left\|\operatorname{div}\left(V^i-\tilde V^i\right)(s)\right\|_{\mathbf{L}^\infty(\Omega\times B(0,K\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)});\mathbb{R})}\mathrm{d}s\\
&\quad+K\|\rho_o\|^2_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\int_0^t\left\|\nabla_w\left(V^i-\tilde V^i\right)(s)\right\|_{\mathbf{L}^\infty(\Omega\times B(0,K\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)});\mathbb{R}^{N\times m})}\mathrm{d}s\\
&\le b(t)\int_o^t\|\rho(s)-\tilde\rho(s)\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\,\mathrm{d}s+c(t)\int_0^t\left\|V(s)-\tilde V(s)\right\|_{\mathbf{C}^1(\Omega\times B(0,K\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)});\mathbb{R}^{nN})}\mathrm{d}s\,,
\end{aligned}
$$

where we denote

$$
\begin{aligned}
a(t)=&\ \exp\left(t\,\mathcal{V}\left(1+K\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\right)\right)\Bigg[\mathcal{O}(1)\|\rho_o\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)}+\mathrm{TV}(\rho_o)\\
&\qquad+t\,\mathcal{V}\,\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\left(1+\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\left(K+\mathcal{K}\Big(\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\Big)+K^2\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\right)\right)\Bigg]\\
b(t)=&\ \mathcal{V}\,K\,a(t)+\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\,\mathcal{V}\left(K+\mathcal{K}\Big(\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\Big)\right)\\
c(t)=&\ a(t)+\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\left(1+K\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\right).
\end{aligned}
$$

Applying Gronwall Lemma to the resulting inequality

$$\begin{aligned}\left\|\rho(t)-\tilde\rho(t)\right\|_{\mathbf{L}^1(\Omega,\mathbb{R}^n)} \le{}& b(t)\int_o^t \left\|\rho(s)-\tilde\rho(s)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\mathrm{d}s\\ &+c(t)\int_0^t \left\|V(s)-\tilde V(s)\right\|_{\mathbf{C}^1(\Omega\times B(0,K\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)});\mathbb{R}^{nN})}\mathrm{d}s\end{aligned}$$

yields

$$\begin{aligned}\left\|\rho(t)-\tilde\rho(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} \le{}& c(t)\int_0^t \left\|V(s)-\tilde V(s)\right\|_{\mathbf{C}^1(\Omega\times B(0,K\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R})});\mathbb{R}^{nN})}\mathrm{d}s\\ &+b(t)\,e^{\int_0^t b(s)\mathrm{d}s}\int_0^t c(s)\,e^{-\int_0^s b(\tau)\mathrm{d}\tau}\,\mathrm{d}s\,.\end{aligned}$$

Since $e^{-\int_0^t b(\tau)\mathrm{d}\tau}+b(t)\int_0^t e^{-\int_0^s b(\tau)\mathrm{d}\tau}\,\mathrm{d}s \le \frac{b(t)}{b(0)}$ we get

$$\left\|\rho(t)-\tilde\rho(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)} \le c(t)\frac{b(t)}{b(0)}e^{t\,b(t)}\int_0^t \left\|V(s)-\tilde V(s)\right\|_{\mathbf{C}^1(\Omega\times B(0,K\|\rho_o\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)});\mathbb{R}^{nN})}\mathrm{d}s \tag{4.33}$$

completing the proof. □

5 Proofs Related to Section 3

Lemma 5.1. *Let Ω and η satisfy* **(Ω)** *and* **(η)**, *with $r_\Omega \le \ell_\eta/4$. Then, the function z defined in* (3.2) *satisfies:*

(***z*.1**) *There exists a $c \in \,]0,1[$, depending only on Ω and on η, such that $z(\Omega)\subseteq[c,1]$.*

(***z*.2**) *$z\in\mathbf{C}^2(\Omega;\mathbb{R})$ and $\nabla z(x)=\int_\Omega \nabla\eta(x-y)\,\mathrm{d}y$, $\nabla^2 z(x)=\int_\Omega\nabla^2\eta(x-y)\,\mathrm{d}y$.*

(***z*.3**) *For all $x\in\Omega$ such that $B(x,\ell_\eta)\subseteq\Omega$, $z(x)=1$.*

Proof. Consider first (***z*.1**). For all $x\in\Omega$ such that $B(x,\ell_\eta/2)\subseteq\Omega$, we have

$$z(x)=\int_\Omega \eta(x-y)\,\mathrm{d}y \ge \int_{B(x,\ell_\eta/2)}\eta(x-y)\,\mathrm{d}y \ge \int_{B(x,r_\Omega)}\eta(x-y)\,\mathrm{d}y = \int_{B(0,r_\Omega)}\eta(-y)\,\mathrm{d}y\ .$$

If on the other hand $B(x,\ell_\eta/2)$ is not contained in Ω, then there exists a $\xi\in B(x,\ell_\eta/2)\cap\partial\Omega$. Call x_ξ a point such that $\xi\in\partial B(x_\xi,r_\Omega)$ and $B(x_\xi,r_\Omega)\subseteq\Omega$, which exists by the interior sphere condition, ensured by **(η)**. Then, for all $y\in B(x_\xi,r_\Omega)$, we have

$$\|y-x\| \le \left\|y-x_\xi\right\|+\left\|x_\xi-\xi\right\|+\|\xi-x\| \le 2\,r_\Omega+\frac{1}{2}\,\ell_\eta \le \ell_\eta$$

showing that $B(x_\xi,r_\Omega)\subseteq B(x,\ell_\eta)$, so that $B(x_\xi-x,r_\Omega)\subseteq B(0,\ell_\eta)$ and

$$z(x)=\int_\Omega\eta(x-y)\,\mathrm{d}y \ge \int_{B(x_\xi,r_\Omega)}\eta(x-y)\,\mathrm{d}y = \int_{B(x_\xi-x,r_\Omega)}\eta(-y)\,\mathrm{d}y\ .$$

In both cases, applying Weiestraß Theorem to the continuous map $\alpha\to\int_{B(\alpha,r_\Omega)}\eta(-y)\,\mathrm{d}y$, for all $x\in\Omega$ we obtain

$$\begin{aligned} z(x) \ge{}& \inf_{\alpha:\ B(\alpha,r_\Omega)\subseteq B(0,\ell_\eta)}\int_{B(\alpha,r_\Omega)}\eta(-y)\,\mathrm{d}y\\ ={}& \inf_{\alpha\in B(0,\ell_\eta-r_\Omega)}\int_{B(\alpha,r_\Omega)}\eta(-y)\,\mathrm{d}y \;=\; \min_{\alpha\in B(0,\ell_\eta-r_\Omega)}\int_{B(\alpha,r_\Omega)}\eta(-y)\,\mathrm{d}y\ .\end{aligned}$$

Define now $c = \min_{\alpha \in B(0,\ell_\eta - r_\Omega)} \int_{B(\alpha, r_\Omega)} \eta(-y)\, \mathrm{d}y$: note that this quantity is strictly positive and strictly less than 1 by **($\boldsymbol{\eta}$)**. The proof of ($\boldsymbol{z}$.**1**) is completed.

The proof of ($\boldsymbol{z}$.**2**) follows noting that $z = \chi_\Omega * \eta$, applying the usual properties of the convolution: $\nabla z = \nabla(\chi_\Omega * \eta) = \chi_\Omega * \nabla z$ and a similar computation yields $\nabla^2 z$.

The property ($\boldsymbol{z}$.**3**) is immediate. □

Proof of Lemma 3.1. The $\mathbf{C}^2$ regularity follows from the standard properties of the convolution product and from Lemma 5.1. The lower and upper bounds on $\rho *_\Omega \eta$ are immediate. For the latter one, for instance, $(\rho *_\Omega \eta)(x) \leq \dfrac{1}{z(x)} \left(\operatorname{ess\,sup}_{B(x,\ell_\eta)\cap\Omega} \rho\right) \int_\Omega \eta(x-y)\, \mathrm{d}y = \operatorname{ess\,sup}_{B(x,\ell_\eta)\cap\Omega} \rho$, completing the proof. □

Proof of Lemma 3.2. With reference to the notation in Section 2, set $N = 2$, $n = 1$, $m = 3$. Call $\boldsymbol{i}$, respectively $\boldsymbol{j}$, a unit vector directed along the x_1, respectively x_2, axis. Define

$$V(t,x,A) = v(A_1) \left(w(x) - \beta\, \frac{A_2\, \boldsymbol{i} + A_3\, \boldsymbol{j}}{\sqrt{1 + {A_2}^2 + {A_3}^2}} \right) \qquad \text{with} \quad \mathcal{J}(\rho) = \begin{bmatrix} \rho *_\Omega \eta_1 \\ \partial_1(\rho *_\Omega \eta_2) \\ \partial_2(\rho *_\Omega \eta_2) \end{bmatrix}.$$

Clearly, $V \in \mathbf{C}^2(\Omega \times \mathbb{R}^3; \mathbb{R}^2)$. The $\mathbf{C}^2$ boundedness of V follows from that of v, from that of w, from that of the map $(A_2, A_3) \to \frac{A_2\, \boldsymbol{i} + A_3\, \boldsymbol{j}}{\sqrt{1+{A_2}^2+{A_3}^2}}$ and from the compactness of $\overline{\Omega}$. Hence, **(V)** holds.

Concerning **(J)**, the $\mathbf{C}^2$ regularity follows from **($\boldsymbol{\eta}$)**, from Lemma 5.1 and from the assumption $\eta_2 \in \mathbf{C}^3$. To prove **(J.1)**, with the notation in Lemma 5.1, consider the different components of $\mathcal{J}$ separately. Recall that $z = \chi_\Omega * \eta$ and write the first component of $\mathcal{J}\rho$ as $\rho *_\Omega \eta_1 = \Big((\rho\, \chi_\Omega) * \eta\Big)/z$:

$$\begin{aligned}
\|\rho *_\Omega \eta_1\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} &\leq \frac{\|\eta_1\|_{\mathbf{L}^\infty(\mathbb{R}^2;\mathbb{R})}}{c}\, \|\rho\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \\
\nabla(\rho *_\Omega \eta_1) &= \frac{1}{z} \left((\rho\chi_\Omega) * \nabla\eta_1 \right) - \frac{\chi_\Omega * \nabla\eta_1}{z^2} \left((\rho\chi_\Omega) * \eta_1 \right) \\
\left\|\nabla(\rho *_\Omega \eta_1)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^2)} &\leq \left(\frac{\|\nabla\eta_1\|_{\mathbf{L}^\infty(\mathbb{R}^2;\mathbb{R}^2)}}{c} + \frac{\|\nabla\eta_1\|_{\mathbf{L}^1(\mathbb{R}^2;\mathbb{R}^2)}\, \|\eta_1\|_{\mathbf{L}^\infty(\mathbb{R}^2;\mathbb{R})}}{c^2} \right) \|\rho\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \\
\nabla^2(\rho *_\Omega \eta_1) &= \frac{1}{z} \left((\rho\chi_\Omega) * \nabla^2\eta_1 \right) - 2\, \frac{\chi_\Omega * \nabla\eta_1}{z^2} \left((\rho\chi_\Omega) * \nabla\eta_1 \right) \\
&\quad - \left((\rho\chi_\Omega) * \eta_1 \right) \left(\frac{\chi_\Omega * \nabla^2\eta_1}{z^2} - \frac{2}{z^3} \left(\chi_\Omega * \nabla\eta_1 \right) \otimes \left(\chi_\Omega * \nabla\eta_1 \right) \right) \\
\left\|\nabla^2(\rho *_\Omega \eta_1)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{2\times 2})} &\leq \Bigg[\frac{\left\|\nabla^2\eta_1\right\|_{\mathbf{L}^\infty(\mathbb{R}^2;\mathbb{R}^{2\times 2})}}{c} + \frac{\|\nabla\eta_1\|_{\mathbf{L}^1(\mathbb{R}^2;\mathbb{R}^2)}\, \|\nabla\eta_1\|_{\mathbf{L}^\infty(\mathbb{R}^2;\mathbb{R}^2)}}{c^2} \\
&\quad + \frac{\|\eta_1\|_{\mathbf{L}^\infty(\mathbb{R}^2;\mathbb{R})}}{c^2} \left(\left\|\nabla^2\eta_1\right\|_{\mathbf{L}^1(\mathbb{R}^2;\mathbb{R}^{2\times 2})} + \frac{2\|\nabla\eta_1\|^2_{\mathbf{L}^1(\mathbb{R}^2;\mathbb{R}^2)}}{c} \right) \Bigg] \|\rho\|_{\mathbf{L}^1(\Omega;\mathbb{R})}.
\end{aligned}$$

The estimates of $\partial_j(\rho *_\Omega \eta_2)$ and $\nabla\partial_j(\rho *_\Omega \eta_2)$, for $j = 1, 2$, are entirely analogous. We only check

$$\nabla^2\partial_j(\rho *_\Omega \eta_2) = \frac{1}{z} \left((\rho\chi_\Omega) * \nabla^2\partial_j\eta_2 \right) - \frac{\chi_\Omega * \partial_j\eta_2}{z^2} \left((\rho\chi_\Omega) * \nabla^2\eta_2 \right)$$

$$
\begin{aligned}
&- 2\,\frac{\chi_\Omega * \nabla\eta_2}{z^2}\Big((\rho\chi_\Omega) * \nabla\partial_j\eta_2\Big) - 2\,\frac{\chi_\Omega * \nabla\partial_j\eta_2}{z^2}\Big((\rho\chi_\Omega) * \nabla\eta_2\Big)\\
&+ \frac{4}{z^3}\Big(\chi_\Omega * \nabla\eta_2\Big)\Big(\chi_\Omega * \partial_j\eta_2\Big)\Big((\rho\chi_\Omega) * \nabla\eta_2\Big)\\
&- \frac{\chi_\Omega * \nabla^2\eta_2}{z^2}\Big((\rho\chi_\Omega) * \partial_j\eta_2\Big) + \frac{2}{z^3}\Big(\chi_\Omega * \nabla^2\eta_2\Big)\Big(\chi_\Omega * \partial_j\eta_2\Big)\Big((\rho\chi_\Omega) * \eta_2\Big)\\
&- \frac{\chi_\Omega * \nabla^2\partial_j\eta_2}{z^2}\Big((\rho\chi_\Omega) * \eta_2\Big) + \frac{2}{z^3}\Big(\chi_\Omega * \nabla\eta_2\Big)\otimes\Big(\chi_\Omega * \nabla\eta_2\Big)\Big((\rho\chi_\Omega) * \partial_j\eta_2\Big)\\
&- 6\,\frac{\chi_\Omega * \partial_j\eta_2}{z^4}\Big(\chi_\Omega * \nabla\eta_2\Big)\otimes\Big(\chi_\Omega * \nabla\eta_2\Big)\Big((\rho\chi_\Omega) * \eta_2\Big)\\
&+ \frac{4}{z^3}\Big(\chi_\Omega * \nabla\eta_2\Big)\Big(\chi_\Omega * \nabla\partial_j\eta_2\Big)\Big((\rho\chi_\Omega) * \eta_2\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\left\|\nabla^2\partial_j(\rho *_\Omega \eta_2)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{2\times 2})}\\
\le &\left(\frac{\left\|\nabla^2\partial_j\eta_2\right\|_{\mathbf{L}^\infty(\mathbb{R}^2;\mathbb{R}^{2\times 2})}}{c} + \frac{\left\|\partial_j\eta_2\right\|_{\mathbf{L}^1(\mathbb{R}^2;\mathbb{R})}\left\|\nabla^2\eta_2\right\|_{\mathbf{L}^\infty(\mathbb{R}^2;\mathbb{R}^{2\times 2})}}{c^2}\right.\\
&+ 2\,\frac{\left\|\nabla\eta_2\right\|_{\mathbf{L}^1(\mathbb{R}^2;\mathbb{R}^2)}\left\|\nabla\partial_j\eta_2\right\|_{\mathbf{L}^\infty(\mathbb{R}^2;\mathbb{R}^2)}}{c^2} + 2\,\frac{\left\|\nabla\partial_j\eta_2\right\|_{\mathbf{L}^1(\mathbb{R}^2;\mathbb{R}^2)}\left\|\nabla\eta_2\right\|_{\mathbf{L}^\infty(\mathbb{R}^2;\mathbb{R}^2)}}{c^2}\\
&+ 4\,\frac{\left\|\nabla\eta_2\right\|_{\mathbf{L}^1(\mathbb{R}^2;\mathbb{R}^2)}\left\|\partial_j\eta_2\right\|_{\mathbf{L}^1(\mathbb{R}^2;\mathbb{R})}\left\|\nabla\eta_2\right\|_{\mathbf{L}^\infty(\mathbb{R}^2;\mathbb{R}^2)}}{c^3} + \frac{\left\|\nabla^2\eta_2\right\|_{\mathbf{L}^1(\mathbb{R}^2;\mathbb{R}^{2\times 2})}\left\|\partial_j\eta_2\right\|_{\mathbf{L}^\infty(\mathbb{R}^2;\mathbb{R})}}{c^2}\\
&+ 2\,\frac{\left\|\nabla^2\eta_2\right\|_{\mathbf{L}^1(\mathbb{R}^2;\mathbb{R}^{2\times 2})}\left\|\partial_j\eta_2\right\|_{\mathbf{L}^1(\mathbb{R}^2;\mathbb{R})}\left\|\eta_2\right\|_{\mathbf{L}^\infty(\mathbb{R}^2;\mathbb{R})}}{c^3} + \frac{\left\|\nabla^2\partial_j\eta_2\right\|_{\mathbf{L}^1(\mathbb{R}^2;\mathbb{R}^{2\times 2})}\left\|\eta_2\right\|_{\mathbf{L}^\infty(\mathbb{R}^2;\mathbb{R})}}{c^2}\\
&+ 2\,\frac{\left\|\nabla\eta_2\right\|^2_{\mathbf{L}^1(\mathbb{R}^2;\mathbb{R}^2)}\left\|\partial_j\eta_2\right\|_{\mathbf{L}^\infty(\mathbb{R}^2;\mathbb{R})}}{c^3} + 6\,\frac{\left\|\partial_j\eta_2\right\|_{\mathbf{L}^1(\mathbb{R}^2;\mathbb{R})}\left\|\nabla^2\eta_2\right\|^2_{\mathbf{L}^1(\mathbb{R}^2;\mathbb{R}^2)}\left\|\eta_2\right\|_{\mathbf{L}^\infty(\mathbb{R}^2;\mathbb{R})}}{c^4}\\
&\left.+4\,\frac{\left\|\nabla\eta_2\right\|_{\mathbf{L}^1(\mathbb{R}^2;\mathbb{R}^2)}\left\|\nabla\partial_j\eta_2\right\|_{\mathbf{L}^1(\mathbb{R}^2;\mathbb{R}^2)}\left\|\eta_2\right\|_{\mathbf{L}^\infty(\mathbb{R}^2;\mathbb{R})}}{c^3}\right)\left\|\rho\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}.
\end{aligned}
$$

Finally, **(J.2)** is now immediate thanks to the linearity of $\mathcal{J}$. □

Proof of Lemma 3.3. Note that (3.5) fits into (1.1) setting $N = 2$, $n = 2$, $m = 10$ and

$$
\begin{aligned}
V^1(t,x,A) &= v^1(A_1)\left(w^1(x) - \frac{\beta_{11}(A_3\boldsymbol{i}+A_4\boldsymbol{j})}{\sqrt{1+{A_3}^2+{A_4}^2}} - \frac{\beta_{12}(A_5\boldsymbol{i}+A_6\boldsymbol{j})}{\sqrt{1+{A_5}^2+{A_6}^2}}\right),\\
V^2(t,x,A) &= v^2(A_2)\left(w^2(x) - \frac{\beta_{21}(A_7\boldsymbol{i}+A_8\boldsymbol{j})}{\sqrt{1+{A_7}^2+{A_8}^2}} - \frac{\beta_{22}(A_9\boldsymbol{i}+A_{10}\boldsymbol{j})}{\sqrt{1+{A_9}^2+{A_{10}}^2}}\right),\\
\mathcal{J}(\rho)_1 = (\rho_1+\rho_2) *_\Omega \eta_1^{11}\,, \quad & \mathcal{J}(\rho)_{3,4} = \nabla_x(\rho_1 *_\Omega \eta_2^{11})\,, \quad \mathcal{J}(\rho)_{5,6} = \nabla_x(\rho_1 *_\Omega \eta_2^{12})\,,\\
\mathcal{J}(\rho)_2 = (\rho_1+\rho_2) *_\Omega \eta_1^{22}\,, \quad & \mathcal{J}(\rho)_{7,8} = \nabla_x(\rho_1 *_\Omega \eta_2^{21})\,, \quad \mathcal{J}(\rho)_{9,10} = \nabla_x(\rho_1 *_\Omega \eta_2^{22})\,,
\end{aligned}
\tag{5.1}
$$

where $\nabla_x = [\partial_1 \quad \partial_2]$. The same computations as in the proof of Lemma 3.2 show that **(V)** and **(J)** hold, completing the proof. □

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